Introduction

Objective

Going into AP calculus, there are certain skills that have been taught to you over the previous years that you must have. If you do not have these skills, you will find that you will consistently get problems incorrect next year, even though you understand the calculus concepts. It is frustrating for students when they are tripped up by the algebra or trigonometry and not the calculus. This summer packet is intended for you to brush up and possibly relearn these topics.

Groundwork

Google Classroom

We'll be using Google Classroom

MathXL for School

1. Registering for MathXL for School

   Before you begin, make sure you have:
   - (a) Your school email address
   - (b) The Temporary Access Code (one word, do NOT type the dashes)

     HSMXLT-GIGLI-BAUTH-SKEAN-LOBBY-TOTES
   - (c) Your Course ID is XL0B-K1RH-7023-6J43
2. To Register, go to www.MathXLforSchool.com and click the Student button under Register.

   Then, follow the instructions on the screen. Create your username & password

   Username: ________________________________

   Password: ________________________________

3. Enrolling in the AP Calculus course:

   After registering, go to MathXLforSchool.com and log in with your username and password.

   Enter the Course ID: XL0B-K1RH-7023-6J43

   After enrolling in your course, you are ready to start working in MathXL for School

4. For additional help with MathXL for School, refer to these resources
   - a. Download a step-by-step visual walkthrough on registering and enrolling: http://www.mathxlforschool.com/support/marketing/student-visual-walkthrough
   - b. Watch a video on how to register and enroll: https://www.mathxlforschool.com/support/marketing/how-to-videos/studentregistration
   - c. Get additional help: http://www.mathxlforschool.com/support/marketing/getting-started-students/

Prerequisites for AP Calculus

Before studying calculus, all students should have successfully completed the equivalent of four years of secondary mathematics designed for college-bound students: courses which should prepare them with a strong foundation in reasoning with algebraic symbols and working with algebraic structures, and trigonometry.

Prospective calculus students should take courses in which they study algebra, geometry, trigonometry, analytic geometry, and elementary functions. These functions include linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and piecewise-defined functions.

Before studying calculus, students must be familiar with the properties of functions, the composition of functions, the algebra of functions, and the graphs of functions. Students must also understand the language of functions (domain and range, odd and even, periodic, symmetry, zeros, intercepts, and descriptors such as increasing and decreasing).

Students should also know how the sine and cosine functions are defined from the unit circle and know the values of the trigonometric functions at the numbers $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples.

In addition students must know the facts regarding the inverse trigonometric functions.

Additional Online Resources:

http://www.purplemath.com/modules/index.htm
https://www.khanacademy.org/math/trigonometry
https://www.mathway.com/Algebra
http://www.sosmath.com/
https://photomath.net/en/
http://www.livemath.com/
https://sites.google.com/a/epsne.org/mr-smith/how-to-study-math
PreCalculus Videos

Summer Packet Outline Due August 5, 2022:

PreRequisites for AP Calculus: Total Number of Exercises: 117

Chapter 0: Prereuisites for Calculus

0.1 Lines (24 Exercises)

Quick Review

1. Increments
2. Slope of a Line
3. Parallel & Perpendicular Lines
4. Equations of Lines
5. Applications

0.2 Functions & Graphs (30 Exercises)

Quick Review

1. Functions
2. Domains & Ranges
3. Viewing & Interpreting Graphs
4. Even Functions & Odd Functions
5. Symmetry
6. Functions Defined in Pieces
7. Absolute Value Function
8. Composite Functions

0.3 Exponential Functions (22 Exercises)

Quick Review

1. Exponential Growth
2. Exponential Decay
3. Applications
4. The Number e

0.4 Parametric Equations (Skip)

0.5 Functions & Logarithms (27 Exercises)

Quick Review

1. One-to-One Functions
2. Inverses
3. Finding Inverses
4. Logarithmic Functions
5. Properties of Logarithms
6. Applications

0.6 Trigonometric Functions (24 Exercises)

Quick Review

1. Radian Measure
2. Graphs of Trigonometric Functions
3. Periodicity
4. Even & Odd Trigonometric Functions
5. Transformations of Trigonometric Graphs
6. Inverse Trigonometric Functions

Calculus Exercises

Chapter 1: Limits & Continuity

1.1 Rates of Change & Limits

Quick Review

1. Interpret Limits
2. Estimate Limits
3. Limits of Sums, differences, products, quotients & composite functions
4. One-sided Limits
5. The Squeeze Theorem

1.2 Limits Involving Infinity

Quick Review

1. The Squeeze Theorem
2. Asymptotic & Unbounded behavior of Functions
3. End Behavior of Functions

1.3 Continuity

Quick Review

1. Continuity at a Point
2. Types of Discontinuity
3. Sums, Differences, products, Quotients & compositions of Continuous Functions
4. The Intermediate Value Theorem

1.4 Rates of Change, Tangent Lines, & Sensitivity

Quick Review

1. Tangent Lines
2. Slopes of Curves
3. Instantaneous Rate of Change
4. Sensitivity

Chapter 2: Derivatives (Just FYI – not required for Summer work)

Chapter 2: Derivatives

2.1 Derivative of a Function

Quick Review

1. The Meaning of Differentiable
2. Different ways of denoting the derivative of a function
3. Given the graph of f ‘(x), graph f(x)
4. Given the graph of f(x), graph f ‘(x)
5. One-sided Derivatives
6. Graphing the Derivative from Data

2.2 Differentiability

Quick Review

1. Why f ‘(a) may fail to exist
2. Differentiability implies Local Linearity
3. Numerical Derivatives on a Calculator

4. Differentiability Implies Continuity
5. Intermediate Value Theorem for Derivatives

2.3 Rules for Differentiation

Quick Review

1. Power Functions
2. Sum & Difference Rules
3. Product & Quotient Rules
4. Negative Powers of x
5. 2nd & Higher Order Derivatives

2.4 Velocity & Other Rates of Change

Quick Review

1. Instantaneous Rates of Change
2. Motion on a Line
3. Acceleration as the 2nd Derivative
4. Modeling Vertical Motion & Particle Motion
5. The Derivative as a Measure of Sensitivity to Change
6. Marginal Cost & Marginal Revenue

2.5 Derivatives of Trigonometric Functions

Quick Review

1. Derivatives of the Sine & Cosine Functions
2. Modeling Harmonic Motion
3. Jerk as the Derivative of Acceleration
4. Derivatives of the Tangent, Cotangent, Secant, & Cosecant Functions
5. The Derivative as a Measure of Sensitivity to Change
6. Tangent & Normal Lines

Prerequisite Exercises:

Chapter 0: Prerequisites for AP Calculus:

0.1 Lines (24 Exercises) All Quick Review & Exercises # 1, 5, 9, 13, 19, 21, 25, 27, 32, 35, 37, 41, 43, 45

Quick Review

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

1. Find the value of y that corresponds to $x = 3$ in $y = -2 + 4(x - 3)$.

2. Find the value of x that corresponds to $y = 3$ in $y = 3 - 2(x + 1)$.

In Exercises 3 and 4, find the value of m that corresponds to the values of x and y.

3. $x = 5,\quad y = 2,\quad m = \dfrac{y - 3}{x - 4}$

4. $x = -1,\quad y = -3,\quad m = \dfrac{2 - y}{3 - x}$

In Exercises 5 and 6, determine whether the ordered pair is a solution to the equation.

5. $3x - 4y = 5$
(a) $(2, 1/4)$ **(b)** $(3, -1)$

6. $y = -2x + 5$
(a) $(-1, 7)$ **(b)** $(-2, 1)$

In Exercises 7 and 8, find the distance between the points.

7. $(1, 0),\quad (0, 1)$

8. $(2, 1),\quad (1, -1/3)$

In Exercises 9 and 10, solve for y in terms of x.

9. $4x - 3y = 7$

10. $-2x + 5y = -3$

EXERCISES 0.1 Lines (24 Exercises) All Quick Review & Exercises # 1, 5, 9, 13, 19, 21, 25, 27, 32, 35, 37, 41, 43, 45

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, find the coordinate increments from A to B.

1. $A(1, 2),\ B(-1, -1)$ **2.** $A(-3, 2),\ B(-1, -2)$

3. $A(-3, 1),\ B(-8, 1)$ **4.** $A(0, 4),\ B(0, -2)$

In Exercises 5–8, let L be the line determined by points A and B.

(a) Plot A and B. **(b)** Find the slope of L.

(c) Draw the graph of L.

5. $A(1, -2),\ B(2, 1)$ **6.** $A(-2, -1),\ B(1, -2)$

7. $A(2, 3),\ B(-1, 3)$ **8.** $A(1, 2),\ B(1, -3)$

In Exercises 9–12, you are given a point P on a line with slope m. Find the y-coordinate of the point with the given x-coordinate.

9. $P(3, 5)\quad m = 2\quad x = 4.5$

10. $P(-2, 1)\quad m = 3\quad x = 2$

11. $P(3, 2)\quad m = -3\quad x = 5$

12. $P(-1, -2)\quad m = 0.8\quad x = 1$

In Exercises 13–17, the position d of a bicyclist (measured in kilometers) is a linear function of time t (measured in minutes). At time $t = 6$ minutes, the position is $d = 5$ km. If the bicyclist travels 2 km for every 5 minutes, find the position of the bicyclist at each time t.

13. $t = 8$ minutes **14.** $t = 3$ minutes

15. $t = 12$ minutes **16.** $t = 20$ minutes

17. Find the linear equation that describes the position d of the bicyclist in Exercises 13–16 as a function of time t.

18. ***Club Fees*** A tennis club charges a monthly fee of \$65 and a rate of \$20 for each half-hour of court time. Find the linear equation that gives the total monthly fee F for a club member who accumulates t hours of court time during the month.

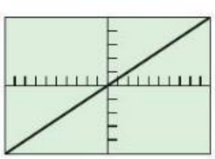
In Exercises 19–22, write the point-slope equation for the line through the point P with slope m.

19. $P(1, 1),\quad m = 1$ **20.** $P(-1, 1),\quad m = -1$

21. $P(0, 3),\quad m = 2$ **22.** $P(-4, 0),\quad m = -2$

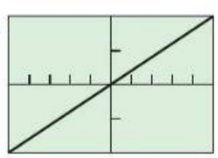
In Exercises 23 and 24, the line contains the origin and the point in the upper right corner of the grapher screen. Write an equation for the line.

23.



[–10, 10] by [–25, 25]

24.



[–5, 5] by [–2, 2]

In Exercises 25–28, find the **(a)** slope and **(b)** y-intercept, and **(c)** graph the line.

25. $3x + 4y = 12$ **26.** $x + y = 2$

27. $\frac{x}{3} + \frac{y}{4} = 1$ **28.** $y = 2x + 4$

In Exercises 29–32, write an equation for the line through P that is **(a)** parallel to L, and **(b)** perpendicular to L.

29. $P(0, 0),\quad L\colon y = -x + 2$ **30.** $P(-2, 2),\quad L\colon 2x + y = 4$

31. $P(-2, 4),\quad L\colon x = 5$ **32.** $P(-1, 1/2),\quad L\colon y = 3$

In Exercises 33–38, find the unique pair (x, y) that satisfies both equations simultaneously.

33. $x - 2y = 13$ and $3x + y = 4$

34. $2x + y = 11$ and $6x - y = 5$

35. $20x + 7y = 22$ and $y - 5x = 11$

36. $2y - 5x = 0$ and $4x + y = 26$

37. $4x - y = 4$ and $14x + 3y = 1$

38. $3x + 2y = 4$ and $12x - 5y = 3$

39. ***Unit Pricing*** If 5 burgers and 4 orders of fries cost \$30.76, while 8 burgers and 6 orders of fries cost \$48.28, what is the cost of a burger and what is the cost of an order of fries?

40. **Writing to Learn** ***x- and y-intercepts***

(a) Explain why c and d are the x-intercept and y-intercept, respectively, of the line

$$\frac{x}{c} + \frac{y}{d} = 1.$$

(b) How are the x-intercept and y-intercept related to c and d in the line

$$\frac{x}{c} + \frac{y}{d} = 2?$$

41. ***Parallel and Perpendicular Lines*** For what value of k are the two lines $2x + ky = 3$ and $x + y = 1$ **(a)** parallel? **(b)** perpendicular?

Group Activity In Exercises 42–44, work in groups of two or three to solve the problem.

42. ***Insulation*** By measuring slopes in the figure below, find the temperature change in degrees per inch for the following materials.

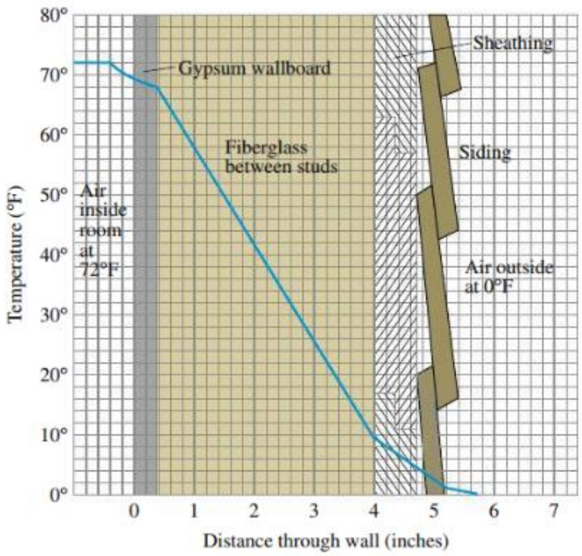
(a) gypsum wallboard

(b) fiberglass insulation

(c) wood sheathing

SECTION 0.1 Lines (24 Exercises) All Quick Review & Exercises # 1, 5, 9, 13, 19, 21, 25, 27, 32, 35, 37, 41, 43, 45

(d) **Writing to Learn** Which of the materials in (a)–(c) is the best insulator? the poorest? Explain.



43. *For the Birds* The level of seed in Bruce's bird feeder declines linearly over time. If the feeder is filled to the 12-inch level at 10:00 AM and is at the 7-inch level at 2:00 PM the same day, at approximately what time will the seed be completely gone?

44. *Modeling Distance Traveled* A car starts from point P at time $t = 0$ and travels at 45 mph.

(a) Write an expression $d(t)$ for the distance the car travels from P.

(b) Graph $y = d(t)$.

(c) What is the slope of the graph in (b)? What does it have to do with the car?

(d) **Writing to Learn** Create a scenario in which t could have negative values.

(e) **Writing to Learn** Create a scenario in which the y-intercept of $y = d(t)$ could be 30.

Standardized Test Questions

45. True or False The slope of a vertical line is zero. Justify your answer.

46. True or False The slope of a line perpendicular to the line $y = mx + b$ is $1/m$. Justify your answer.

47. Multiple Choice Which of the following is an equation of the line through $(-3, 4)$ with slope $1/2$?

(A) $y - 4 = \frac{1}{2}(x + 3)$ (B) $y + 3 = \frac{1}{2}(x - 4)$

(C) $y - 4 = -2(x + 3)$ (D) $y - 4 = 2(x + 3)$

(E) $y + 3 = 2(x - 4)$

48. Multiple Choice Which of the following is an equation of the vertical line through $(-2, 4)$?

(A) $y = 4$ (B) $x = 2$ (C) $y = -4$

(D) $x = 0$ (E) $x = -2$

49. Multiple Choice Which of the following is the x-intercept of the line $y = 2x - 5$?

(A) $x = -5$ (B) $x = 5$ (C) $x = 0$

(D) $x = 5/2$ (E) $x = -5/2$

50. Multiple Choice Which of the following is an equation of the line through $(-2, -1)$ parallel to the line $y = -3x + 1$?

(A) $y = -3x + 5$ (B) $y = -3x - 7$ (C) $y = \frac{1}{3}x - \frac{1}{3}$

(D) $y = -3x + 1$ (E) $y = -3x - 4$

Extending the Ideas

51. *Tangent to a Circle* A circle with radius 5 centered at the origin passes through the point $(3, 4)$. Find an equation for the line that is tangent to the circle at that point.

52. *Knowing Your Rights* The vertices of triangle ABC have coordinates $A(-3, 10)$, $B(1, 3)$, and $C(15, 11)$. Prove that it is a right triangle. Which side is the hypotenuse?

53. *Simultaneous Linear Equations Revisited* The two linear equations shown below are said to be *dependent and inconsistent*:

$$3x - 5y = 3$$
$$-9x + 15y = 8$$

(a) Solve the equations simultaneously by an algebraic method, either substitution or elimination. What is your conclusion?

(b) What happens if you use a graphical method?

(c) **Writing to Learn** Explain in algebraic and graphical terms what happens when two linear equations are dependent and inconsistent.

54. *Simultaneous Linear Equations Revisited Again* The two linear equations shown below are said to be *dependent and consistent*:

$$2x - 5y = 3$$
$$6x - 15y = 9$$

(a) Solve the equations simultaneously by an algebraic method, either substitution or elimination. What is your conclusion?

(b) What happens if you use a graphical method?

(c) **Writing to Learn** Explain in algebraic and graphical terms what happens when two linear equations are dependent and consistent.

SECTION 0.1 0.1 Lines (24 Exercises) All Quick Review & Exercises # 1, 5, 9, 13, 19, 21, 25, 27, 32, 35, 37, 41, 43, 45

55. ***Parallelogram*** Three different parallelograms have vertices at $(-1, 1)$, $(2, 0)$, and $(2, 3)$. Draw the three and give the coordinates of the missing vertices.

56. ***Parallelogram*** Show that if the midpoints of consecutive sides of any quadrilateral are connected, the result is a parallelogram.

57. ***Tangent Line*** Consider the circle of radius 5 centered at $(1, 2)$. Find an equation of the line tangent to the circle at the point $(-2, 6)$.

58. **Group Activity** ***Distance from a Point to a Line*** This activity investigates how to find the distance from a point $P(a, b)$ to a line $L: Ax + By = C$.

(a) Write an equation for the line M through P perpendicular to L.

(b) Find the coordinates of the point Q in which M and L intersect.

(c) Find the distance from P to Q.

0.2 Functions & Graphs (30 Exercises) All of Quick Review and # 2, 6, 7, 9, 15, 16, 22, 25, 28, 29, 31, 33, 37, 38, 43, 47, 52, 55

QUICK REVIEW

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–6, solve for x.

1. $3x - 1 \le 5x + 3$

2. $x(x - 2) > 0$

3. $|x - 3| \le 4$

4. $|x - 2| \ge 5$

5. $x^2 < 16$

6. $9 - x^2 \ge 0$

In Exercises 7 and 8, describe how the graph of f can be transformed to the graph of g.

7. $f(x) = x^2, \quad g(x) = (x + 2)^2 - 3$

8. $f(x) = |x|, \quad g(x) = |x - 5| + 2$

In Exercises 9–12, find all real solutions to the equations.

9. $f(x) = x^2 - 5$

(a) $f(x) = 4$ (b) $f(x) = -6$

10. $f(x) = 1/x$

(a) $f(x) = -5$ (b) $f(x) = 0$

11. $f(x) = \sqrt{x + 7}$

(a) $f(x) = 4$ (b) $f(x) = 1$

12. $f(x) = \sqrt[3]{x - 1}$

(a) $f(x) = -2$ (b) $f(x) = 3$

EXERCISES 0.2:

EXERCISES 0.2: Functions & Graphs (30 Exercises) All of Quick Review and # 2, 6, 7, 9, 15, 16, 22, 25, 28, 29, 31, 33, 37, 38, 43, 47, 52, 55

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, **(a)** write a formula for the function and **(b)** use the formula to find the indicated value of the function.

1. the area A of a circle as a function of its diameter d; the area of a circle of diameter 4 in.

2. the height h of an equilateral triangle as a function of its side length s; the height of an equilateral triangle of side length 3 m

3. the surface area S of a cube as a function of the length of the cube's edge e; the surface area of a cube of edge length 5 ft

4. the volume V of a sphere as a function of the sphere's radius r; the volume of a sphere of radius 3 cm

In Exercises 5–12, **(a)** identify the domain and range and **(b)** sketch the graph of the function.

5. $y = 4 - x^2$ **6.** $y = x^2 - 9$

7. $y = 2 + \sqrt{x - 1}$ **8.** $y = -\sqrt{-x}$

9. $y = \dfrac{1}{x - 2}$ **10.** $y = \sqrt[4]{-x}$

11. $y = 1 + \dfrac{1}{x}$ **12.** $y = 1 + \dfrac{1}{x^2}$

In Exercises 13–20, use a grapher to **(a)** identify the domain and range and **(b)** draw the graph of the function.

13. $y = \sqrt[3]{x}$ **14.** $y = 2\sqrt{3 - x}$

15. $y = \sqrt[3]{1 - x^2}$ **16.** $y = \sqrt{9 - x^2}$

17. $y = x^{2/5}$ **18.** $y = x^{3/2}$

19. $y = \sqrt[3]{x - 3}$ **20.** $y = \dfrac{1}{\sqrt{4 - x^2}}$

In Exercises 21–30, determine whether the function is even, odd, or neither.

21. $y = x^4$ **22.** $y = x + x^2$

23. $y = x + 2$ **24.** $y = x^2 - 3$

25. $y = \sqrt{x^2 + 2}$ **26.** $y = x + x^3$

27. $y = \dfrac{x^3}{x^2 - 1}$ **28.** $y = \sqrt[3]{2 - x}$

29. $y = \dfrac{1}{x - 1}$ **30.** $y = \dfrac{1}{x^2 - 1}$

In Exercises 31–34, graph the piecewise-defined functions.

31. $f(x) = \begin{cases} 3 - x, & x \le 1 \\ 2x, & 1 < x \end{cases}$ **32.** $f(x) = \begin{cases} 1, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

33. $f(x) = \begin{cases} 4 - x^2, & x < 1 \\ (3/2)x + 3/2, & 1 \le x \le 3 \\ x + 3, & x > 3 \end{cases}$

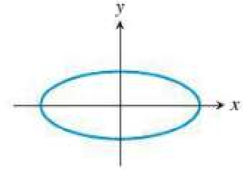
34. $f(x) = \begin{cases} x^2, & x < 0 \\ x^3, & 0 \le x \le 1 \\ 2x - 1, & x > 1 \end{cases}$

35. Writing to Learn The *vertical line test* to determine whether a curve is the graph of a function states: If every vertical line in the xy-plane intersects a given curve in at most one point, then the curve is the graph of a function. Explain why this is true.

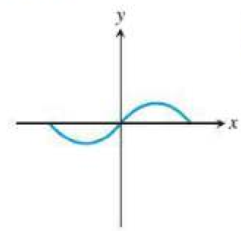
36. Writing to Learn For a curve to be *symmetric about the x-axis,* the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x-axis is not the graph of a function, unless the function is $y = 0$.

In Exercises 37–40, use the vertical line test (see Exercise 35) to determine whether the curve is the graph of a function.

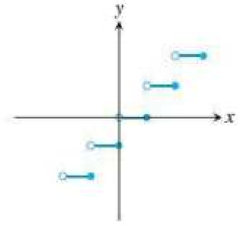
37.



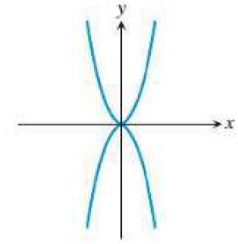
38.



39.

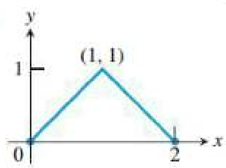


40.

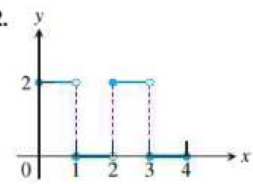


In Exercises 41–48, write a piecewise formula for the function.

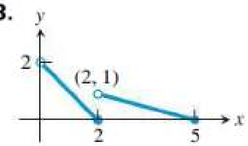
41.



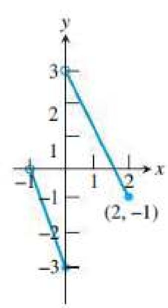
42.



43.

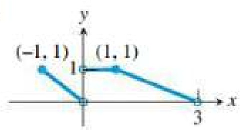


44.

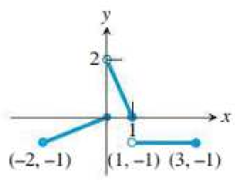


EXERCISES 0.2: Functions & Graphs (30 Exercises) All of Quick Review and # 2, 6, 7, 9, 15, 16, 22, 25, 28, 29, 31, 33, 37, 38, 43, 47, 52, 55

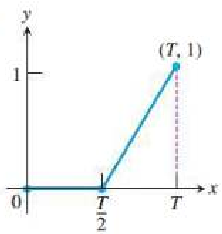
45.



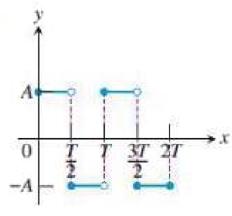
46.



47.



48.



In Exercises 49 and 50, **(a)** draw the graph of the function. Then find its **(b)** domain and **(c)** range.

49. $f(x) = -|3 - x| + 2$ **50.** $f(x) = 2|x + 4| - 3$

In Exercises 51 and 52, find

(a) $f(g(x))$ **(b)** $g(f(x))$ **(c)** $f(g(0))$

(d) $g(f(0))$ **(e)** $g(g(-2))$ **(f)** $f(f(x))$

51. $f(x) = x + 5, \quad g(x) = x^2 - 3$

52. $f(x) = x + 1, \quad g(x) = x - 1$

53. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
(a)	?	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
(b)	?	$1 + 1/x$	x
(c)	$1/x$	?	x
(d)	$\sqrt{x}$	?	$\lvert x\rvert, x \geq 0$

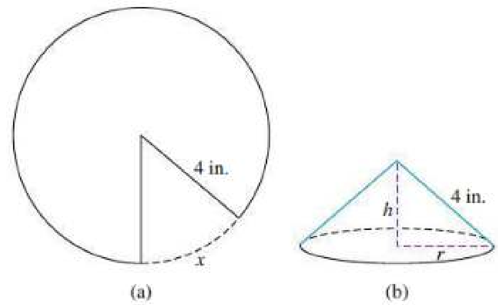
54. ***The Cylindrical Can Problem*** A cylindrical can is to be constructed so that its height h is equal to its diameter.

(a) Write the volume of the cylinder as a function of its height h.

(b) Write the total surface area of the cylindrical can (including the top and bottom) as a function of its height h.

(c) Find the volume of the can if its total surface area is 54π square inches.

55. ***The Cone Problem*** Begin with a circular piece of paper with a 4-in. radius as shown in (a). Cut out a sector with an arc length of x. Join the two edges of the remaining portion to form a cone with radius r and height h, as shown in (b).



(a) Explain why the circumference of the base of the cone is $8\pi - x$.

(b) Express the radius r as a function of x.

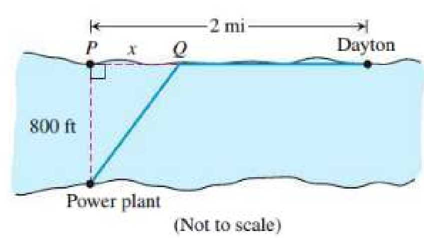
(c) Express the height h as a function of x.

(d) Express the volume V of the cone as a function of x.

56. ***Industrial Costs*** Dayton Power and Light, Inc., has a power plant on the Miami River where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs \$180 per foot across the river and \$100 per foot along the land.

(a) Suppose that the cable goes from the plant to a point Q on the opposite side that is x ft from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x.

(b) Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P.



Standardized Test Questions

57. True or False The function $f(x) = x^4 + x^2 + x$ is an even function. Justify your answer.

58. True or False The function $f(x) = x^{-3}$ is an odd function. Justify your answer.

EXERCISES 0.2: Functions & Graphs (30 Exercises) All of Quick Review and # 2, 6, 7, 9, 15, 16, 22, 25, 28, 29, 31, 33, 37, 38, 43, 47, 52, 55

59. Multiple Choice Which of the following gives the domain of $f(x) = \dfrac{x}{\sqrt{9 - x^2}}$?

(A) $x \neq \pm 3$ **(B)** $(-3, 3)$ **(C)** $[-3, 3]$
(D) $(-\infty, -3) \cup (3, \infty)$ **(E)** $(3, \infty)$

60. Multiple Choice Which of the following gives the range of $f(x) = 1 + \dfrac{1}{x - 1}$?

(A) $(-\infty, 1) \cup (1, \infty)$ **(B)** $x \neq 1$ **(C)** all real numbers
(D) $(-\infty, 0) \cup (0, \infty)$ **(E)** $x \neq 0$

61. Multiple Choice If $f(x) = 2x - 1$ and $g(x) = x + 3$, which of the following gives $(f \circ g)(2)$?

(A) 2 **(B)** 6 **(C)** 7 **(D)** 9 **(E)** 10

62. Multiple Choice The length L of a rectangle is twice as long as its width W. Which of the following gives the area A of the rectangle as a function of its width?

(A) $A(W) = 3W$ **(B)** $A(W) = \frac{1}{2}W^2$
(C) $A(W) = 2W^2$ **(D)** $A(W) = W^2 + 2W$
(E) $A(W) = W^2 - 2W$

Explorations

In Exercises 63–66, **(a)** graph $f \circ g$ and $g \circ f$ and make a conjecture about the domain and range of each function. **(b)** Then confirm your conjectures by finding formulas for $f \circ g$ and $g \circ f$.

63. $f(x) = x - 7, \quad g(x) = \sqrt{x}$

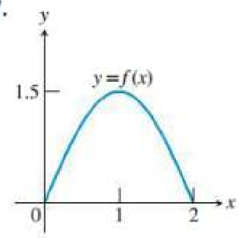
64. $f(x) = 1 - x^2, \quad g(x) = \sqrt{x}$

65. $f(x) = x^2 - 3, \quad g(x) = \sqrt{x + 2}$

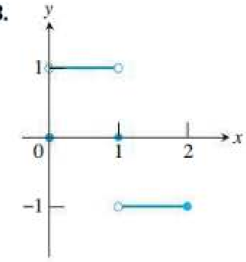
66. $f(x) = \dfrac{2x - 1}{x + 3}, \quad g(x) = \dfrac{3x + 1}{2 - x}$

Group Activity In Exercises 67–70, a portion of the graph of a function defined on $[-2, 2]$ is shown. Complete each graph assuming that the graph is **(a)** even, **(b)** odd.

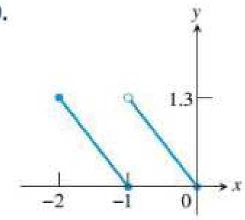
67.



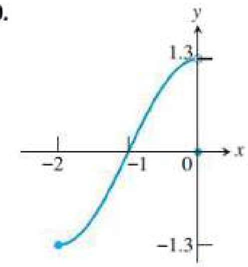
68.



69.



70.



Extending the Ideas

71. Enter $y_1 = \sqrt{x}$, $y_2 = \sqrt{1 - x}$ and $y_3 = y_1 + y_2$ on your grapher.

(a) Graph y_3 in $[-3, 3]$ by $[-1, 3]$.

(b) Compare the domain of the graph of y_3 with the domains of the graphs of y_1 and y_2.

(c) Replace y_3 by

$$y_1 - y_2, \quad y_2 - y_1, \quad y_1 \cdot y_2, \quad y_1/y_2, \quad \text{and} \quad y_2/y_1,$$

in turn, and repeat the comparison of part (b).

(d) Based on your observations in (b) and (c), what would you conjecture about the domains of sums, differences, products, and quotients of functions?

72. ***Even and Odd Functions***

(a) Must the product of two even functions always be even? Give reasons for your answer.

(b) Can anything be said about the product of two odd functions? Give reasons for your answer.

0.3 Exponential Functions (22 Exercises) : All of Quick Review and # 2, 5, 9, 13,17,19,21,22, 25, 31, 33, 38

QUICK REVIEW

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–3, evaluate the expression. Round your answers to 3 decimal places.

1. $5^{2/3}$ **2.** $3^{\sqrt{2}}$

3. $3^{-1.5}$

In Exercises 4–6, solve the equation. Round your answers to 4 decimal places.

4. $x^3 = 17$ **5.** $x^5 = 24$

6. $x^{10} = 1.4567$

In Exercises 7 and 8, find the value of investing P dollars for n years with the interest rate r compounded annually.

7. $P = \$500, \quad r = 4.75\%, \quad n = 5$ years

8. $P = \$1000, \quad r = 6.3\%, \quad n = 3$ years

In Exercises 9 and 10, simplify the exponential expression.

9. $\dfrac{(x^{-3}y^2)^2}{(x^4y^3)^3}$ **10.** $\left(\dfrac{a^3b^{-2}}{c^4}\right)^2\left(\dfrac{a^4c^{-2}}{b^3}\right)^{-1}$

EXERCISES 0.3

EXERCISES 0.3 Exponential Functions (22 Exercises) : All of Quick Review and # 2, 5, 9, 13,17,19,21,22, 25, 31, 33, 38

In Exercises 9–12, use a graph to find the zeros of the function.

9. $f(x) = 2^x - 5$

10. $f(x) = e^x - 4$

11. $f(x) = 3^x - 0.5$

12. $f(x) = 3 - 2^x$

In Exercises 13–18, match the function with its graph. Try to do it without using your grapher.

13. $y = 2^x$

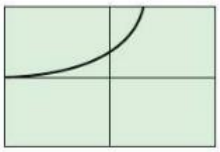
14. $y = 3^{-x}$

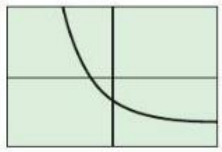
15. $y = -3^{-x}$

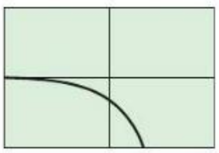
16. $y = -0.5^{-x}$

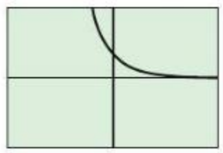
17. $y = 2^{-x} - 2$

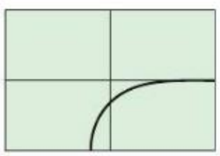
18. $y = 1.5^x - 2$

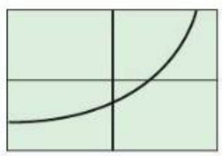

(a)


(b)


(c)


(d)


(e)


(f)

In Exercises 19–32, use an exponential model to solve the problem.

19. *Population Growth* The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?

20. *Population Growth* The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

(a) Estimate the population in 1915 and 1940.

(b) Approximately when did the population reach 50,000?

21. *Half-life* The approximate half-life of titanium-44 is 63 years. How long will it take a sample to lose

(a) 50% of its titanium-44?

(b) 75% of its titanium-44?

22. *Half-life* The amount of silicon-32 in a sample will decay from 28 grams to 7 grams in approximately 340 years. What is the approximate half-life of silicon-32?

23. *Radioactive Decay* The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.

(a) Express the amount of phosphorus-32 remaining as a function of time t.

(b) When will there be 1 gram remaining?

24. *Finding Time* If John invests \$2300 in a savings account with a 6% interest rate compounded annually, how long will it take until John's account has a balance of \$4150?

25. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

26. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded monthly.

27. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded continuously.

28. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded annually.

29. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded daily.

30. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.

31. *Cholera Bacteria* Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 h?

32. *Eliminating a Disease* Suppose that in any given year, the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

(a) to reduce the number of cases to 1000?

(b) to eliminate the disease; that is, to reduce the number of cases to less than 1?

Group Activity In Exercises 33–36, copy and complete the table for the function.

33. $y = 2x - 3$

x	y	Change (Δy)
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

34. $y = -3x + 4$

x	y	Change (Δy)
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

35. $y = x^2$

x	y	Change (Δy)
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

EXERCISES 0.3 Exponential Functions (22 Exercises) : All of Quick Review and # 2, 5, 9, 13,17,19,21,22, 25, 31, 33, 38

36. $y = 3e^x$

x	y	Ratio (y_i/y_{i-1})
1	?	?
2	?	?
3	?	?
4	?	

37. Writing to Learn Explain how the change Δy is related to the slopes of the lines in Exercises 33 and 34. If the changes in x are constant for a linear function, what would you conclude about the corresponding changes in y?

38. *Bacteria Growth* The number of bacteria in a petri dish culture after t hours is

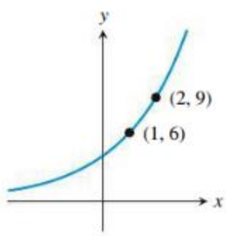
$$B = 100e^{0.693t}.$$

(a) What was the initial number of bacteria present?

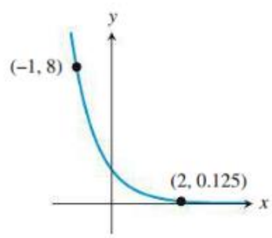
(b) How many bacteria are present after 6 hours?

(c) Approximately when will the number of bacteria be 200? Estimate the doubling time of the bacteria.

39. The graph of the exponential function $y = a \cdot b^x$ is shown below. Find a and b.



40. The graph of the exponential function $y = a \cdot b^x$ is shown below. Find a and b.



Standardized Test Questions

You may use a graphing calculator to solve the following problems.

41. True or False The number 3^{-2} is negative. Justify your answer.

42. True or False If $4^3 = 2^a$, then $a = 6$. Justify your answer.

43. Multiple Choice John invests \$200 at 4.5% compounded annually. About how long will it take for John's investment to double in value?

(A) 6 yr **(B)** 9 yr **(C)** 12 yr **(D)** 16 yr **(E)** 20 yr

44. Multiple Choice Which of the following gives the domain of $y = 2e^{-x} - 3$?

(A) $(-\infty, \infty)$ **(B)** $[-3, \infty)$ **(C)** $[-1, \infty)$ **(D)** $(-\infty, 3]$ **(E)** $x \neq 0$

45. Multiple Choice Which of the following gives the range of $y = 4 - 2^{2x}$?

(A) $(-\infty, \infty)$ **(B)** $(-\infty, 4)$ **(C)** $[-4, \infty)$ **(D)** $(-\infty, 4]$ **(E)** all reals

46. Multiple Choice Which of the following gives the best approximation for the zero of $f(x) = 4 - e^x$?

(A) $x = -1.386$ **(B)** $x = 0.386$ **(C)** $x = 1.386$ **(D)** $x = 3$ **(E)** There are no zeros.

Exploration

47. Let $y_1 = x^2$ and $y_2 = 2^x$.

(a) Graph y_1 and y_2 in $[-5, 5]$ by $[-2, 10]$. How many times do you think the two graphs cross?

(b) Compare the corresponding changes in y_1 and y_2 as x changes from 1 to 2, 2 to 3, and so on. How large must x be for the changes in y_2 to overtake the changes in y_1?

(c) Solve for x: $x^2 = 2^x$.

(d) Solve for x: $x^2 < 2^x$.

Extending the Ideas

In Exercises 48 and 49, assume that the graph of the exponential function $f(x) = k \cdot a^x$ passes through the two points. Find the values of a and k.

48. $(1, 4.5), (-1, 0.5)$

49. $(1, 1.5), (-1, 6)$

0.5 Functions & Logarithms All of Quick Review Plus # 1, 2, 3, 5, 7, 9, 13, 15, 19, 23, 25, 29, 33, 35, 37, 41, 47

QUICK REVIEW 0.5

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, let $f(x) = \sqrt[3]{x-1}$, $g(x) = x^2 + 1$, and evaluate the expression.

1. $(f \circ g)(1)$

2. $(g \circ f)(-7)$

3. $(f \circ g)(x)$

4. $(g \circ f)(x)$

In Exercises 5 and 6, choose parametric equations and a parameter interval to represent the function on the interval specified.

5. $y = \dfrac{1}{x-1}, \quad x \geq 2$ **6.** $y = x, \quad x < -3$

In Exercises 7–10, find the points of intersection of the two curves. Round your answers to 2 decimal places.

7. $y = 2x - 3, \quad y = 5$

8. $y = -3x + 5, \quad y = -3$

9. **(a)** $y = 2^x, \quad y = 3$

(b) $y = 2^x, \quad y = -1$

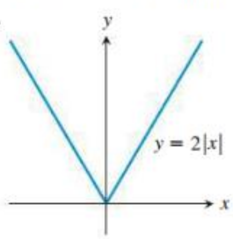
10. **(a)** $y = e^{-x}, \quad y = 4$

(b) $y = e^{-x}, \quad y = -1$

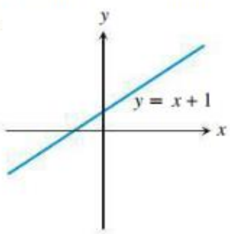
EXERCISES 0.5 All of Quick Review Plus # 1, 2, 3, 5, 7, 9, 13, 15, 19, 23, 25, 29, 33, 35, 37, 41, 47

In Exercises 1–6, determine whether the function is one-to-one.

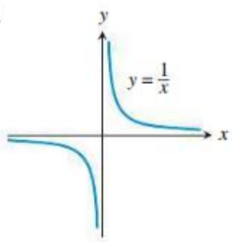
1.



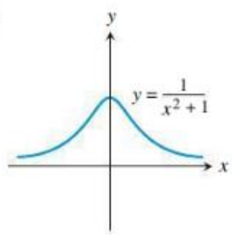
2.



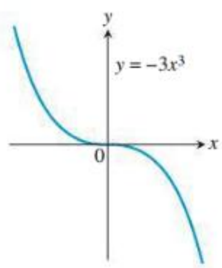
3.



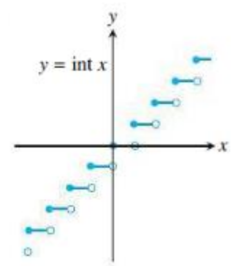
4.



5.



6.



In Exercises 7–12, determine whether the function has an inverse function.

7. $y = \dfrac{3}{x-2} - 1$

8. $y = x^2 + 5x$

9. $y = x^3 - 4x + 6$

10. $y = x^3 + x$

11. $y = \ln x^2$

12. $y = 2^{3-x}$

EXERCISES 0.5 All of Quick Review Plus # 1, 2, 3, 5, 7, 9, 13, 15, 19, 23, 25, 29, 33, 35, 37, 41, 47

In Exercises 13–24, find f^{-1} and verify that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x.$$

13. $f(x) = 2x + 3$ **14.** $f(x) = 5 - 4x$

15. $f(x) = x^3 - 1$ **16.** $f(x) = x^2 + 1, \quad x \geq 0$

17. $f(x) = x^2, \quad x \leq 0$ **18.** $f(x) = x^{2/3}, \quad x \geq 0$

19. $f(x) = -(x - 2)^2, \quad x \leq 2$

20. $f(x) = x^2 + 2x + 1, \quad x \geq -1$

21. $f(x) = \dfrac{1}{x^2}, \quad x > 0$ **22.** $f(x) = \dfrac{1}{x^3}$

23. $f(x) = \dfrac{2x + 1}{x + 3}$ **24.** $f(x) = \dfrac{x + 3}{x - 2}$

In Exercises 25–32, use parametric graphing to graph f, f^{-1}, and $y = x$.

25. $f(x) = e^x$ **26.** $f(x) = 3^x$ **27.** $f(x) = 2^{-x}$

28. $f(x) = 3^{-x}$ **29.** $f(x) = \ln x$ **30.** $f(x) = \log x$

31. $f(x) = \sin^{-1} x$ **32.** $f(x) = \tan^{-1} x$

In Exercises 33–36, solve the equation algebraically. You can check your solution graphically.

33. $(1.045)^t = 2$ **34.** $e^{0.05t} = 3$

35. $e^x + e^{-x} = 3$ **36.** $2^x + 2^{-x} = 5$

In Exercises 37 and 38, solve for y.

37. $\ln y = 2t + 4$ **38.** $\ln(y - 1) - \ln 2 = x + \ln x$

In Exercises 39–42, draw the graph and determine the domain and range of the function.

39. $y = 2\ln(3 - x) - 4$ **40.** $y = -3\log(x + 2) + 1$

41. $y = \log_2(x + 1)$ **42.** $y = \log_3(x - 4)$

In Exercises 43 and 44, find a formula for f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

43. $f(x) = \dfrac{100}{1 + 2^{-x}}$ **44.** $f(x) = \dfrac{50}{1 + 1.1^{-x}}$

45. ***Self-inverse*** Prove that the function f is its own inverse.

(a) $f(x) = \sqrt{1 - x^2}, \quad x \geq 0$ **(b)** $f(x) = 1/x$

46. ***Radioactive Decay*** The half-life of a certain radioactive substance is 12 hours. There are 8 grams present initially.

(a) Express the amount of substance remaining as a function of time t.

(b) When will there be 1 gram remaining?

47. ***Doubling Your Money*** Determine how much time is required for a \$500 investment to double in value if interest is earned at the rate of 4.75% compounded annually.

48. ***Population Growth*** The population of Glenbrook is 375,000 and is increasing at the rate of 2.25% per year. Predict when the population will be 1 million.

49. ***Guess the Curve*** A curve is defined parametrically as the set of points $(\sqrt{2 - t}, \sqrt{2 + t})$ for $-2 \leq t \leq 2$. Answer parts (a) through (d) before using your grapher.

(a) Explain why this parametrization cannot be used for other values of t.

(b) If a point is on this curve, what is its distance from the origin?

(c) Find the endpoints of the curve (determined by $t = -2$ and $t = 2$, respectively).

(d) Explain why all other points on this curve must lie in the first quadrant.

(e) Based on what you know from (a) through (d), give a complete geometric description of the curve. Then verify your answer with your grapher.

50. ***Logarithmic Equations*** For an algebraic challenge, solve these equations *without a calculator* by using the Laws of Logarithms.

(a) $4 \ln \sqrt{e^x} = 26$

(b) $x - \log(100) = \ln(e^3)$

(c) $\log_x(7x - 10) = 2$

(d) $2\log_3 x - \log_3(x - 2) = 2$

51. **Group Activity** ***Inverse Functions*** Let $y = f(x) = mx + b$, $m \neq 0$.

(a) **Writing to Learn** Give a convincing argument that f is a one-to-one function.

(b) Find a formula for the inverse of f. How are the slopes of f and f^{-1} related?

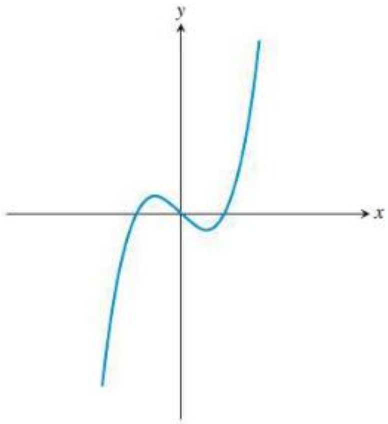
(c) If the graphs of two functions are parallel lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

(d) If the graphs of two functions are perpendicular lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

EXERCISE 0.5 All of Quick Review Plus # 1, 2, 3, 5, 7, 9, 13, 15, 19, 23, 25, 29, 33, 35, 37, 41, 47

Standardized Test Questions

52. True or False The function displayed in the graph below is one-to-one. Justify your answer.



53. True or False If $(f \circ g)(x) = x$, then g is the inverse function of f. Justify your answer.

In Exercises 54 and 55, use the function $f(x) = 3 - \ln(x + 2)$.

54. Multiple Choice Which of the following is the domain of f?

(A) $x \neq -2$ **(B)** $(-\infty, \infty)$ **(C)** $(-2, \infty)$
(D) $[-1.9, \infty)$ **(E)** $(0, \infty)$

55. Multiple Choice Which of the following is the range of f?

(A) $(-\infty, \infty)$ **(B)** $(-\infty, 0)$ **(C)** $(-2, \infty)$
(D) $(0, \infty)$ **(E)** $(0, 5.3)$

56. Multiple Choice Which of the following is the inverse of $f(x) = 3x - 2$?

(A) $g(x) = \dfrac{1}{3x - 2}$ **(B)** $g(x) = x$ **(C)** $g(x) = 3x - 2$
(D) $g(x) = \dfrac{x - 2}{3}$ **(E)** $g(x) = \dfrac{x + 2}{3}$

57. Multiple Choice Which of the following is a solution of the equation $2 - 3^{-x} = -1$?

(A) $x = -2$ **(B)** $x = -1$ **(C)** $x = 0$
(D) $x = 1$ **(E)** There are no solutions.

Exploration

58. *Supporting the Quotient Rule* Let $y_1 = \ln(x/a)$, $y_2 = \ln x$, $y_3 = y_2 - y_1$, and $y_4 = e^{y_3}$.

(a) Graph y_1 and y_2 for $a = 2, 3, 4$, and 5. How are the graphs of y_1 and y_2 related?

(b) Graph y_3 for $a = 2, 3, 4$, and 5. Describe the graphs.

(c) Graph y_4 for $a = 2, 3, 4$, and 5. Compare the graphs to the graph of $y = a$.

(d) Use $e^{y_3} = e^{y_2 - y_1} = a$ to solve for y_1.

Extending the Ideas

59. *One-to-One Functions* If f is a one-to-one function, prove that $g(x) = -f(x)$ is also one-to-one.

60. *One-to-One Functions* If f is a one-to-one function and $f(x)$ is never zero, prove that $g(x) = 1/f(x)$ is also one-to-one.

61. *Domain and Range* Suppose that $a \neq 0$, $b \neq 1$, and $b > 0$. Determine the domain and range of the function.

(a) $y = a(b^{c-x}) + d$

(b) $y = a \log_b(x - c) + d$

62. Group Activity *Inverse Functions*

Let $f(x) = \dfrac{ax + b}{cx + d}$, $c \neq 0$, $ad - bc \neq 0$.

(a) Writing to Learn Give a convincing argument that f is one-to-one.

(b) Find a formula for the inverse of f.

(c) Find the horizontal and vertical asymptotes of f.

(d) Find the horizontal and vertical asymptotes of f^{-1}. How are they related to those of f?

0.6 Trigonometric Functions (24 Exercises)

All of Quick Review Plus # 1,3, 7, 9, 14, 16, 17, 22, 28, 32, 35, 39, 43, 43

0.6 QUICK REVIEW:

Exercise numbers with a gray background indicate problems that the authors have designed to be solved without a calculator.

In Exercises 1–4, convert from radians to degrees or degrees to radians.

1. $\pi/3$ **2.** -2.5 **3.** $-40°$ **4.** $45°$

In Exercises 5–7, solve the equation graphically in the given interval.

5. $\sin x = 0.6$, $0 \leq x \leq 2\pi$ **6.** $\cos x = -0.4$, $0 \leq x \leq 2\pi$

7. $\tan x = 1$, $-\dfrac{\pi}{2} \leq x < \dfrac{3\pi}{2}$

8. Show that $f(x) = 2x^2 - 3$ is an even function. Explain wh its graph is symmetric about the y-axis.

9. Show that $f(x) = x^3 - 3x$ is an odd function. Explain why its graph is symmetric about the origin.

10. Give one way to restrict the domain of the function $f(x) = x^4 - 2$ to make the resulting function one-to-one.

0.6 EXERCISES All of Quick Review Plus # 1,3, 7, 9, 14, 16, 17, 22, 28, 32, 35, 39, 43, 43

In Exercises 1–4, the angle lies at the center of a circle and subtends an arc of the circle. Find the missing angle measure, circle radius, or arc length.

Angle	Radius	Arc Length
1. $5\pi/8$	2	?
2. 175°	?	10
3. ?	14	7
4. ?	6	$3\pi/2$

In Exercises 5–8, determine if the function is even or odd.

5. secant

6. tangent

7. cosecant

8. cotangent

In Exercises 9 and 10, find all the trigonometric values of θ with the given conditions.

9. $\cos\theta = -\frac{15}{17}, \quad \sin\theta > 0$

10. $\tan\theta = -1, \quad \sin\theta < 0$

0.6 EXERCISES All of Quick Review Plus # 1,3, 7, 9, 14, 16, 17, 22, 28, 32, 35, 39, 43, 43

In Exercises 11–14, determine **(a)** the period, **(b)** the domain, **(c)** the range, and **(d)** draw the graph of the function.

11. $y = 3\csc(3x + \pi) - 2$ **12.** $y = 2\sin(4x + \pi) + 3$

13. $y = -3\tan(3x + \pi) + 2$

14. $y = 2\sin\left(2x + \frac{\pi}{3}\right)$

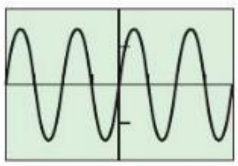
In Exercises 15 and 16, choose an appropriate viewing window to display two complete periods of each trigonometric function in radian mode.

15. **(a)** $y = \sec x$ **(b)** $y = \csc x$ **(c)** $y = \cot x$

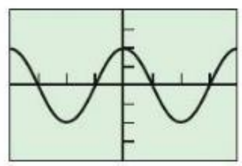
16. **(a)** $y = \sin x$ **(b)** $y = \cos x$ **(c)** $y = \tan x$

In Exercises 17–22, specify **(a)** the period, **(b)** the amplitude, and **(c)** identify the viewing window that is shown. [Caution: Do not assume that the tick marks on both axes are at integer values.]

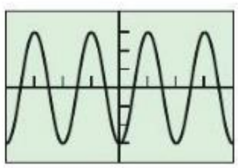
17. $y = 1.5\sin 2x$



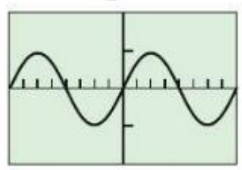
18. $y = 2\cos 3x$



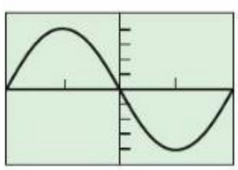
19. $y = -3\cos 2x$



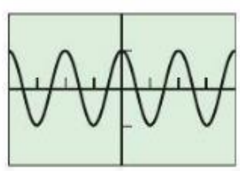
20. $y = 5\sin\frac{x}{2}$



21. $y = -4\sin\frac{\pi}{3}x$



22. $y = \cos \pi x$



23. The frequencies for the seven "white key" notes produced on the tempered scale of a piano (starting with middle C) are shown in Table 1.4. A computer analyzes the pressure displacement versus time for the wave produced by a tuning fork and gives its equation as $y = 1.23\sin(2073.55x - 0.49) + 0.44$.

(a) Estimate the frequency of the note produced by the tuning fork.

(b) Identify the note produced by the tuning fork.

TABLE 1.4 Frequencies of Musical Notes

C	D	E	F	G	A	B
262	294	330	349	392	440	494

24. ***Temperature Data*** Table 1.5 gives the average monthly temperatures for St. Louis for a 12-month period starting with January. Model the monthly temperature with an equation of the form

$$y = a\sin[b(t - h)] + k,$$

with y in degrees Fahrenheit, t in months, as follows:

TABLE 1.5
Temperature Data for St. Louis

Time (months)	Temperature (°F)
1	34
2	30
3	39
4	44
5	58
6	67
7	78
8	80
9	72
10	63
11	51
12	40

(a) Find the value of b, assuming that the period is 12 months.

(b) How is the amplitude a related to the difference $80° - 30°$?

(c) Use the information in (b) to find k.

(d) Find h, and write an equation for y.

(e) Superimpose a graph of y on a scatter plot of the data.

In Exercises 25–26, show that the function is one-to-one, and graph its inverse.

25. $y = \cos x,\quad 0 \le x \le \pi$ **26.** $y = \tan x,\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

In Exercises 27–30, give the measure of the angle in radians and degrees. Give exact answers whenever possible.

27. $\sin^{-1}(0.5)$ **28.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

29. $\tan^{-1}(-5)$ **30.** $\cos^{-1}(0.7)$

In Exercises 31–36, solve the equation in the specified interval.

31. $\tan x = 2.5,\quad 0 \le x \le 2\pi$

32. $\cos x = -0.7,\quad 2\pi \le x < 4\pi$

33. $\csc x = 2,\quad 0 < x < 2\pi$ **34.** $\sec x = -3,\quad -\pi \le x < \pi$

35. $\sin x = -0.5,\quad -\infty < x < \infty$

36. $\cot x = -1,\quad -\infty < x < \infty$

0.6 EXERCISES All of Quick Review Plus # 1,3, 7, 9, 14, 16, 17, 22, 28, 32, 35, 39, 43, 43

In Exercises 37–40, use the given information to find the values of the six trigonometric functions at the angle θ. Give exact answers.

37. $\theta = \sin^{-1}\left(\frac{8}{17}\right)$

38. $\theta = \tan^{-1}\left(-\frac{5}{12}\right)$

39. The point $P(-3, 4)$ is on the terminal side of θ.

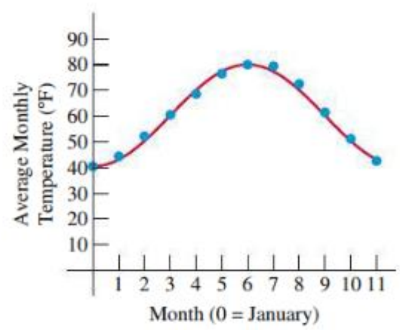
40. The point $P(-2, 2)$ is on the terminal side of θ.

In Exercises 41 and 42, evaluate the expression.

41. $\sin\left(\cos^{-1}\left(\frac{7}{11}\right)\right)$

42. $\tan\left(\sin^{-1}\left(\frac{9}{13}\right)\right)$

43. ***Chattanooga Temperatures*** The average monthly temperatures in Chattanooga, TN, range from a low of 40.5°F in January to a high of 80.0°F in July. Setting January as month 0 and December as month 11, the temperature cycle can be nicely modeled by a sinusoid with equation $y = A\cos(Bx) + C$, as shown in the graph below. Find the values of A, B, and C. [*Source: www.weatherbase.com*]



44. ***Rocky Mountain Highs*** The average monthly high temperatures in Steamboat Springs, CO, are shown in Table 1.6 below. Setting January as month 0 and December as month 11, construct a sinusoid with equation $y = A\cos(Bx) + C$ that models the temperature cycle in Steamboat Springs. Support your answer with a graph and a scatter plot on your calculator. [*Source: www.weatherbase.com*]

TABLE 1.6
Average Monthly Highs in Steamboat Springs, CO

JAN	FEB	MAR	APR	MAY	JUN
28.9	33.8	42.0	53.6	65.4	75.5
JUL	AUG	SEP	OCT	NOV	DEC
82.6	80.3	72.5	60.4	43.3	30.7

45. ***Even-Odd***

(a) Show that $\cot x$ is an odd function of x.

(b) Show that the quotient of an even function and an odd function is an odd function.

46. ***Even-Odd***

(a) Show that $\csc x$ is an odd function of x.

(b) Show that the reciprocal of an odd function is odd.

47. ***Even-Odd*** Show that the product of an even function and an odd function is an odd function.

48. ***Finding the Period*** Give a convincing argument that the period of $\tan x$ is π.

49. ***Is the Product of Sinusoids a Sinusoid?*** Make a conjecture, and then use your graphing calculator to support your answers to the following questions.

(a) Is the product $y = (\sin x)(\sin 2x)$ a sinusoid? What is the period of the function?

(b) Is the product $y = (\sin x)(\cos x)$ a sinusoid? What is the period of the function?

(c) One of the functions in (a) or (b) above can be written in the form $y = A\sin(Bx)$. Identify the function and find A and B.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

50. True or False The period of $y = \sin(x/2)$ is π. Justify your answer.

51. True or False The amplitude of $y = \frac{1}{2}\cos x$ is 1. Justify your answer.

In Exercises 52–54, $f(x) = 2\cos(4x + \pi) - 1$.

52. Multiple Choice Which of the following is the domain of f?

(A) $[-\pi, \pi]$ **(B)** $[-3, 1]$ **(C)** $[-1, 4]$ **(D)** $(-\infty, \infty)$ **(E)** $x \neq 0$

53. Multiple Choice Which of the following is the range of f?

(A) $(-3, 1)$ **(B)** $[-3, 1]$ **(C)** $(-1, 4)$ **(D)** $[-1, 4]$ **(E)** $(-\infty, \infty)$

54. Multiple Choice Which of the following is the period of f?

(A) 4π **(B)** 3π **(C)** 2π **(D)** π **(E)** $\pi/2$

55. Multiple Choice Which of the following is the measure of $\tan^{-1}(-\sqrt{3})$ in degrees?

(A) $-60°$ **(B)** $-30°$ **(C)** $30°$ **(D)** $60°$ **(E)** $120°$

Exploration

56. ***Trigonometric Identities*** Let $f(x) = \sin x + \cos x$.

(a) Graph $y = f(x)$. Describe the graph.

(b) Use the graph to identify the amplitude, period, horizontal shift, and vertical shift.

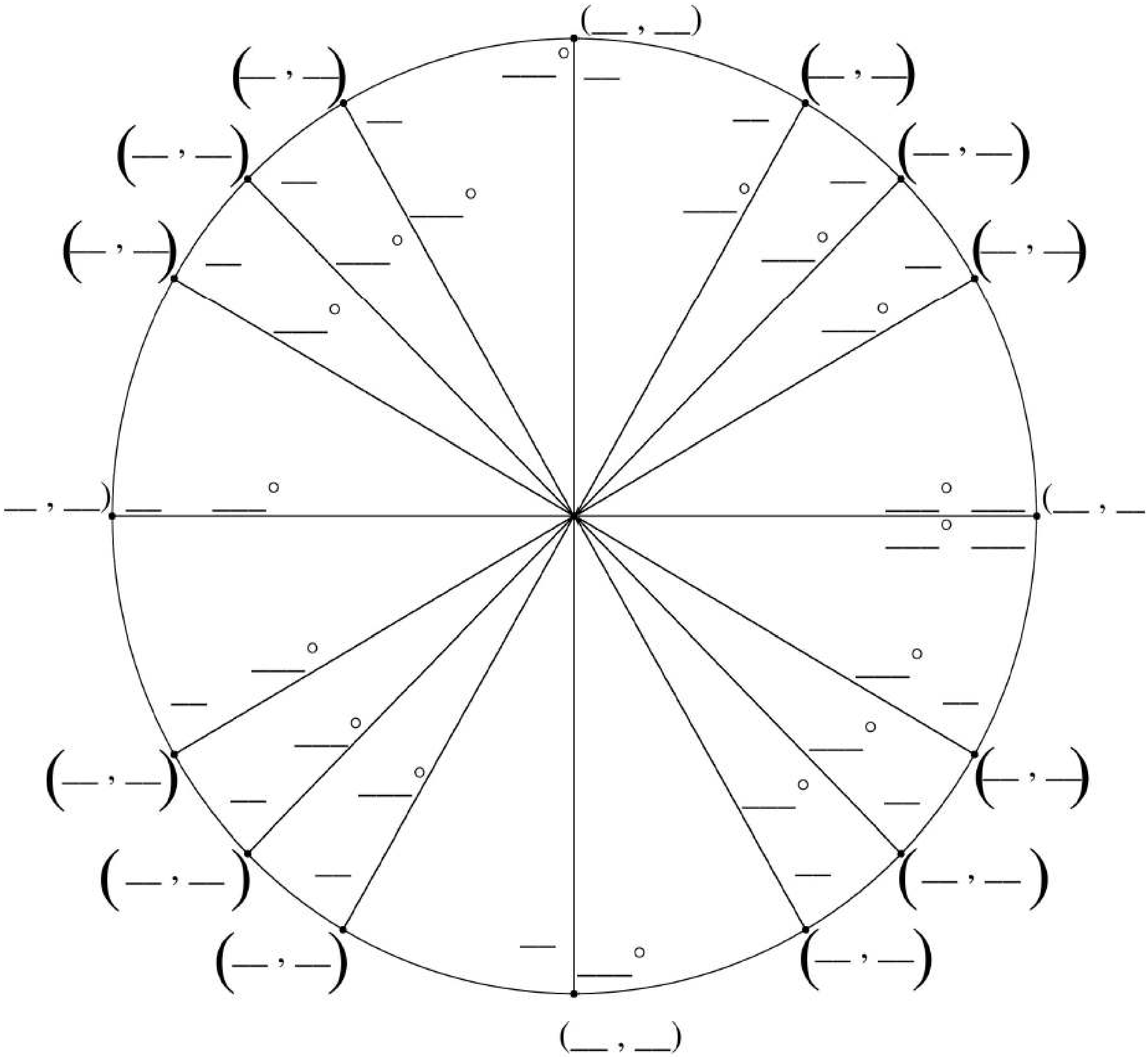
(c) Use the formula

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$$

for the sine of the sum of two angles to confirm your answers.

YOU MUST KNOW THE UNIT CIRCLE

Fill in The Unit Circle



Calculus Reading & Exercises:

Chapter 1: Limits & Continuity

1.1 Rates of Change & Limits

Chapter 1 Overview

As shown in the Prologue, calculus looks for values that familiar techniques from algebra can only approximate. The actual value that we seek is the number that lies between all pairs of bounds, no matter how close. This value is called the *limit,* a concept that is a fundamental building block of calculus.

In this chapter, we show how to define and calculate limits of function values. The calculation rules are straightforward, and most of the limits we need can be found by substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

One of the uses of limits lies in building a careful definition of continuity. Continuous functions arise frequently in scientific work because they model such an enormous range of natural behavior and because they have special mathematical properties.

1.1 Rates of Change and Limits

You will be able to interpret, estimate, and determine limits of function values.

- Interpretation and expression of limits using correct notation
- Estimation of limits using numerical and graphical information
- Limits of sums, differences, products, quotients, and composite functions
- Interpretation and expression of one-sided limits
- The Squeeze Theorem

Average and Instantaneous Velocity

The average velocity of a moving body during an interval of time is found by dividing the change in distance or position by the change in time. More precisely, if $y = f(t)$ is a distance or position function of a moving body at time t, then the **average rate of change** (or **average velocity**) is the ratio

$$\frac{\Delta y}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t},$$

where the elapsed time is the interval from t to $t + \Delta t$, or simply Δt, and the distance traveled during this time interval is $f(t + \Delta t) - f(t)$. It is also common to use the letter h instead of Δt to denote the elapsed time, in which case the average rate of change can be written

$$\frac{\Delta y}{\Delta t} = \frac{f(t + h) - f(t)}{h}.$$

Example 1 Finding an Average Velocity

A rock breaks loose from the top of a tall cliff. What is its average velocity during the first 2 seconds of fall?

Solution

Experiments show that a dense solid object dropped from rest to fall freely near the surface of the earth will fall

$$y = 16t^2$$

feet in the first t seconds. The average velocity of the rock over any given time interval is the distance traveled, Δy, divided by the length of the interval Δt. For the first 2 seconds of fall, from $t = 0$ to $t = 2$, we have

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}.$$

Now Try Exercise 1.

Free Fall

Near the surface of the earth, all bodies fall with the same constant acceleration. The distance a body falls after it is released from rest is a constant multiple of the square of the time fallen. At least, that is what happens when a body falls in a vacuum, where there is no air to slow it down. The square-of-time rule also holds for dense, heavy objects like rocks, ball bearings, and steel tools during the first few seconds of fall through air, before the velocity builds up to where air resistance begins to matter. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls *free fall.*

The velocity of a falling rock is always increasing. If we know the position as a function of time, we can calculate average velocity over any given interval of time. But we can also talk about its **instantaneous velocity** or **instantaneous rate of change,** the velocity at one instant of time. As we will see after the next example, we need the idea of *limit* to make precise what we mean by instantaneous rate of change.

Example 2 Finding an Instantaneous Velocity

Find the velocity of the rock in Example 1 at the instant $t = 2$.

Solution

We can calculate the average velocity of the rock over the interval from time $t = 2$ to any slightly later time $t = 2 + h$ as

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}. \quad (1)$$

The average velocity from time $t = 2 - h$ to time $t = 2$ is

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(2-h)^2}{h}. \quad (2)$$

We cannot use either of these formulas to calculate the velocity at the exact instant $t = 2$ because that would require taking $h = 0$, and $0/0$ is undefined. However, we can use these expressions to put bounds on the velocity at $t = 2$.

If we expand the numerator in Equation 1 and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h} = \frac{64h + 16h^2}{h} = 64 + 16h.$$

Similarly, for the average velocity before $t = 2$, Equation 2 simplifies to

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(2-h)^2}{h} = 64 - 16h.$$

The velocity at the instant $t = 2$ lies between $64 - 16h$ and $64 + 16h$ (Table 1.1). The only number that lies between these two bounds for every positive value of h is 64. The limit is 64 ft/sec. We say that the limit of the average velocity as h approaches 0 is 64 ft/sec. *Now Try Exercise 3.*

Formal Definition of Limit

The formal definition of a limit is given in Appendix A, pp. 583–589. This appendix also illustrates how the formal definition is applied and how it leads to the *Properties of Limits* given in Theorem 1.

TABLE 1.1 Average Velocities over Short Time Intervals Near $t = 2$

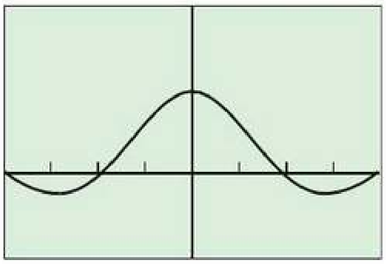
$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(2-h)^2}{h} \quad \text{or} \quad \frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

Length of Time Interval h (sec)	Average Velocity Before $t = 2$, $\frac{\Delta y}{\Delta t}$ (ft/sec)	Average Velocity After $t = 2$, $\frac{\Delta y}{\Delta t}$ (ft/sec)
1	48	80
0.1	62.4	65.6
0.01	63.84	64.16
0.001	63.984	64.016
0.0001	63.9984	64.0016
0.00001	63.99984	64.00016

Definition of Limit

As in the preceding example, limits that give us the instantaneous rate of change can be viewed as numerical limits of values of functions. And this is where a graphing utility and calculus come in. A calculator can suggest the limits, and calculus can give the mathematics for confirming the limits analytically.

Limits give us a language for describing how the outputs of a function behave as the inputs approach some particular value. In Example 2, the average velocity was not defined at $h = 0$ but approached the limit 64 as h approached 0. We were able to see this numerically and to confirm it algebraically by eliminating h from the denominator. But we cannot



$[-2\pi, 2\pi]$ by $[-1, 2]$

(a)

X	Y1	
−.3	.98507	
−.2	.99335	
−.1	.99833	
0	ERROR	
.1	.99833	
.2	.99335	
.3	.98507	
Y1 = sin(X)/X		

(b)

Figure 1.1 (a) A graph and (b) table of values for $f(x) = (\sin x)/x$ that suggest the limit of f as x approaches 0 is 1.

always do that. For instance, we can see both graphically and numerically (Figure 1.1) that the values of $f(x) = (\sin x)/x$ are bounded above by 1 and approach 1 as x approaches 0.

We cannot eliminate the x from the denominator of $(\sin x)/x$ to confirm the observation algebraically. We need to use a theorem about limits to make that confirmation, as you will see in Exercise 77.

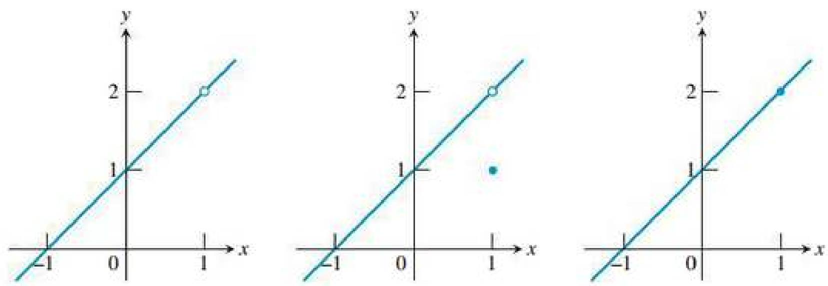
The sentence $\lim_{x \to c} f(x) = L$ is read, "The limit of f of x as x approaches c equals L." The notation means that we can force $f(x)$ to be as close to L as we wish simply by restricting the distance between x and c, *but not allowing x to equal c.*

We saw in Example 2 that $\lim_{h \to 0} (64 + 16h) = 64$.

As suggested in Figure 1.1,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Because we need to distinguish between what happens at c and what happens near c, the value or existence of the limit as $x \to c$ *never* depends on how the function may or may not be defined at c. This is illustrated in Figure 1.2. The function f has limit 2 as $x \to 1$ even though f is not defined at 1. The function g has limit 2 as $x \to 1$ even though $g(1) \neq 2$. The function h is the only one whose limit as $x \to 1$ equals its value at $x = 1$.



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

Figure 1.2 $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2.$

Properties of Limits

By applying six basic facts about limits, we can calculate many unfamiliar limits from limits we already know. For instance, from knowing that

$$\lim_{x \to c} (k) = k \quad \text{Limit of the function with constant value } k$$

and

$$\lim_{x \to c} (x) = c, \quad \text{Limit of the identity function at } x = c$$

we can calculate the limits of all polynomial and rational functions. The facts are listed in Theorem 1.

Theorem 1 Properties of Limits

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

continued

2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$

   The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

   The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$

   The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$

   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If r and s are integers, $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s},$$

   provided that $L^{r/s}$ is a real number and $L > 0$ if s is even.

   The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number and $L > 0$ if s is even.

Using Analytic Methods

We remind the student that *unless otherwise stated,* all examples and exercises are to be done using analytic algebraic methods *without* the use of graphing calculators or computer algebra systems.

Here are some examples of how Theorem 1 can be used to find limits of polynomial and rational functions.

Example 3 Using Properties of Limits

Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$, and the properties of limits, to find the following limits.

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$

Solution

(a)
$$\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 \quad \text{Sum and Difference Rules}$$
$$= c^3 + 4c^2 - 3 \quad \text{Power and Constant Multiple Rules}$$

(b)
$$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)} \quad \text{Quotient Rule}$$
$$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} \quad \text{Sum and Difference Rules}$$
$$= \frac{c^4 + c^2 - 1}{c^2 + 5} \quad \text{Power Rule}$$

Now Try Exercises 5 and 6.

Example 3 shows the remarkable strength of Theorem 1. From the two simple observations that $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$, we can immediately work our way to limits of polynomial functions and most rational functions using substitution.

Theorem 2 Polynomial and Rational Functions

1. If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

2. If $f(x)$ and $g(x)$ are polynomials and c is any real number, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad \text{provided that } g(c) \neq 0.$$

Example 4 Using Theorem 2

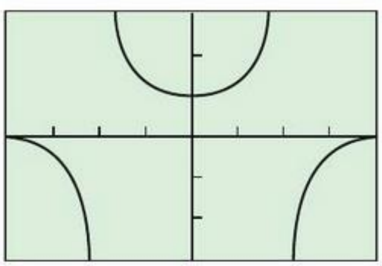
(a) $\lim_{x \to 3} [x^2(2 - x)] = (3)^2(2 - 3) = -9$

(b) $\lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2} = \frac{(2)^2 + 2(2) + 4}{2 + 2} = \frac{12}{4} = 3$ ***Now Try Exercises 9 and 11.***

As with polynomials, limits of many familiar functions can be found by substitution at points where they are defined. This includes trigonometric functions, exponential and logarithmic functions, and composites of these functions. Feel free to use these properties.

Example 5 Using the Product Rule

Determine $\lim_{x \to 0} \frac{\tan x}{x}$.



$[-\pi, \pi]$ by $[-3, 3]$

Figure 1.3 The graph of $f(x) = (\tan x)/x$ suggests that $f(x) \to 1$ as $x \to 0$. (Example 5)

Solution

The graph of $f(x) = (\tan x)/x$ in Figure 1.3 suggests that the limit exists and is about 1. Using the analytic result of Exercise 77, we have

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) \qquad \tan x = \frac{\sin x}{\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} \qquad \text{Product Rule}$$

$$= 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = 1.$$

Now Try Exercise 33.

Sometimes we can use a graph to discover that limits do not exist, as illustrated by Example 6.

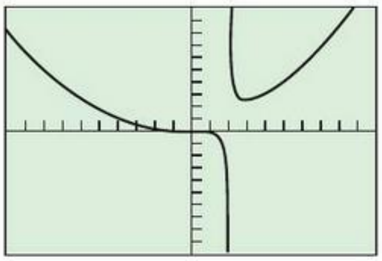
Example 6 Exploring a Nonexistent Limit

Use a graph to explore whether

$$\lim_{x \to 2} \frac{x^3 - 1}{x - 2}$$

exists.

continued



[–10, 10] by [–100, 100]

Figure 1.4 The graph of $f(x) = (x^3 - 1)/(x - 2)$. (Example 6)

Solution

Notice that the denominator is 0 when x is replaced by 2, so we cannot use substitution to determine the limit. The graph in Figure 1.4 of $f(x) = (x^3 - 1)/(x - 2)$ strongly suggests that as $x \to 2$ from either side, the absolute values of the function values get very large. This, in turn, suggests that the limit does not exist.

Now Try Exercise 35.

One-Sided and Two-Sided Limits

Sometimes we need to distinguish between what happens to the function just to the right of c and just to the left. To do this, we call the limit of f as x approaches c from the right the **right-hand limit** of f at c and the limit as x approaches c from the left the **left-hand limit** of f at c. The notation that we use follows.

right-hand: $\lim_{x \to c^+} f(x)$ *The limit of f as x approaches c from the right*

left-hand: $\lim_{x \to c^-} f(x)$ *The limit of f as x approaches c from the left*

Example 7 Function Values Approach Two Numbers

The greatest integer function $f(x) = \text{int } x$ has different right-hand and left-hand limits at each integer, as we can see in Figure 1.5. For example,

$$\lim_{x \to 3^+} \text{int } x = 3 \quad \text{and} \quad \lim_{x \to 3^-} \text{int } x = 2.$$

The limit of int x as x approaches an integer n from the right is n, and the limit as x approaches n from the left is $n - 1$. ***Now Try Exercises 37 and 38.***

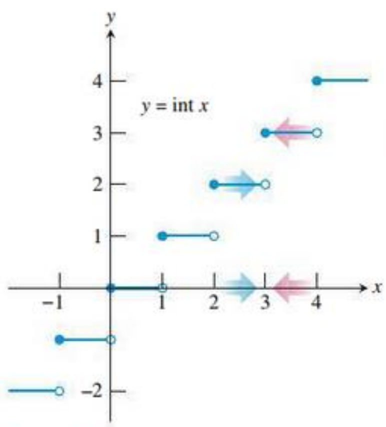


Figure 1.5 At each integer, the greatest integer function $y = \text{int } x$ has different right-hand and left-hand limits. (Example 7)

The greatest integer function, which appears as int x on most calculators, is known to mathematicians as the **floor function,** written $\lfloor x \rfloor$, where we use only the bottom horizontal parts of the brackets to indicate that we go *down* until we reach an integer. You should be able to recognize and use either notation.

We sometimes call $\lim_{x \to c} f(x)$ the **two-sided limit** of f at c to distinguish it from the *one-sided* right-hand and left-hand limits of f at c. Theorem 3 shows how these limits are related.

On the Far Side

If f is not defined to the left of $x = c$, then f does not have a left-hand limit at c. Similarly, if f is not defined to the right of $x = c$, then f does not have a right-hand limit at c.

Theorem 3 One-Sided and Two-Sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.$$

Thus, the greatest integer function $f(x) = \text{int } x$ of Example 7 does not have a limit as $x \to 3$ even though each one-sided limit exists.

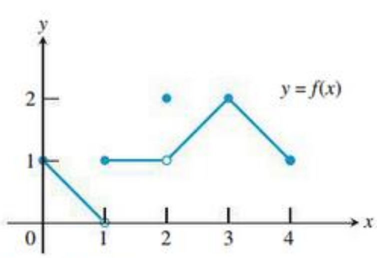


Figure 1.6 The graph of the function

$$f(x) = \begin{cases} -x + 1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x - 1, & 2 < x \le 3 \\ -x + 5, & 3 < x \le 4. \end{cases}$$

(Example 8)

Example 8 Exploring Right- and Left-Hand Limits

All the following statements about the function $y = f(x)$ graphed in Figure 1.6 are true.

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x \to 1^+} f(x) = 1$,

f has no limit as $x \to 1$. (The right- and left-hand limits at 1 are not equal, so $\lim_{x \to 1} f(x)$ does not exist.)

continued

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1,$

$\lim_{x \to 2^+} f(x) = 1,$

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2.$

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 2 = f(3) = \lim_{x \to 3} f(x).$

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1.$

At noninteger values of c between 0 and 4, f has a limit as $x \to c$.

Now Try Exercise 43.

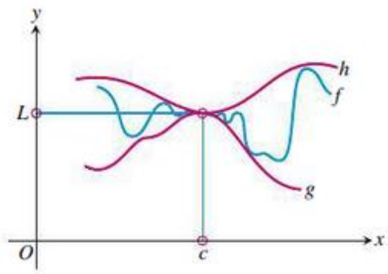


Figure 1.7 Squeezing f between g and h creates a bottleneck around the point (c, L). If we keep x close to c, the bottleneck forces $f(x)$ to be close to L.

Squeeze Theorem

If we cannot find a limit directly, we may be able to find it indirectly with the Squeeze Theorem. The theorem refers to a function f whose values are squeezed between the values of two other functions, g and h. If g and h have the same limit as $x \to c$, then f has that limit too, as suggested by Figure 1.7.

Theorem 4 The Squeeze Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

Example 9 Using the Squeeze Theorem

Show that $\lim_{x \to 0} [x^2 \sin(1/x)] = 0.$

Solution

We know that the values of the sine function lie between -1 and 1. So it follows that

$$\left| x^2 \sin \frac{1}{x} \right| = |x^2| \cdot \left| \sin \frac{1}{x} \right| \le |x^2| \cdot 1 = x^2$$

and

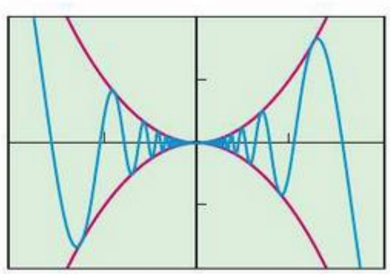
$$-x^2 \le x^2 \sin \frac{1}{x} \le x^2.$$

Because $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$, the Squeeze Theorem gives

$$\lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right) = 0.$$

The graphs in Figure 1.8 support this result.

Now Try Exercise 65.



[−0.2, 0.2] by [−0.02, 0.02]

Figure 1.8 The graphs of $y_1 = x^2$, $y_2 = x^2 \sin(1/x)$, and $y_3 = -x^2$. Notice that $y_3 \le y_2 \le y_1$. (Example 9)

1.1 Work All Quick Review

Quick Review

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, find $f(2)$.

1. $f(x) = 2x^3 - 5x^2 + 4$

2. $f(x) = \dfrac{4x^2 - 5}{x^3 + 4}$

3. $f(x) = \sin\left(\pi \dfrac{x}{2}\right)$

4. $f(x) = \begin{cases} 3x - 1, & x < 2 \\ \dfrac{1}{x^2 - 1}, & x \geq 2 \end{cases}$

In Exercises 5–8, write the inequality in the form $a < x < b$.

5. $|x| < 4$

6. $|x| < c^2$

7. $|x - 2| < 3$

8. $|x - c| < d^2$

In Exercises 9 and 10, write the fraction in reduced form.

9. $\dfrac{x^2 - 3x - 18}{x + 3}$

10. $\dfrac{2x^2 - x}{2x^2 + x - 1}$

1.1 EXERCISES # 1, 3, 5, 11, 17, 23, 35, 37, 43, 45, 51, 55, 57, 65

In Exercises 1–4, an object dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first t seconds.

1. Find the average speed during the first 3 seconds of fall.

2. Find the average speed during the first 4 seconds of fall.

3. Find the speed of the object at $t = 3$ seconds and confirm your answer algebraically.

4. Find the speed of the object at $t = 4$ seconds and confirm your answer algebraically.

1.1 EXERCISES # 1, 3, 5, 11, 17, 23, 35, 37, 43, 45, 51, 55, 57, 65

In Exercises 5 and 6, use $\lim_{x \to c} k = k$, $\lim_{x \to c} x = c$, and the properties of limits to find the limit.

5. $\lim_{x \to c} (2x^3 - 3x^2 + x - 1)$

6. $\lim_{x \to c} \dfrac{x^4 - x^3 + 1}{x^2 + 9}$

In Exercises 7–14, determine the limit by substitution.

7. $\lim_{x \to -1/2} 3x^2(2x - 1)$

8. $\lim_{x \to -4} (x + 3)^{2016}$

9. $\lim_{x \to 1} (x^3 + 3x^2 - 2x - 17)$

10. $\lim_{y \to 2} \dfrac{y^2 + 5y + 6}{y + 2}$

11. $\lim_{y \to -3} \dfrac{y^2 + 4y + 3}{y^2 - 3}$

12. $\lim_{x \to 1/2} \operatorname{int} x$

13. $\lim_{x \to -2} (x - 6)^{2/3}$

14. $\lim_{x \to 2} \sqrt{x + 3}$

In Exercises 15–20, complete the following tables and state what you believe $\lim_{x \to 0} f(x)$ to be.

(a)

x	-0.1	-0.01	-0.001	-0.0001	...
$f(x)$	?	?	?	?	

(b)

x	0.1	0.01	0.001	0.0001	...
$f(x)$	?	?	?	?	

15. $f(x) = \dfrac{x^2 + 6x + 2}{x + 1}$

16. $f(x) = \dfrac{x^2 - x}{x}$

17. $f(x) = x \sin \dfrac{1}{x}$

18. $f(x) = \sin \dfrac{1}{x}$

19. $f(x) = \dfrac{10^x - 1}{x}$

20. $f(x) = x \sin(\ln |x|)$

In Exercises 21–24, explain why you cannot use substitution to determine the limit. Find the limit if it exists.

21. $\lim_{x \to -2} \sqrt{x - 2}$

22. $\lim_{x \to 0} \dfrac{1}{x^2}$

23. $\lim_{x \to 0} \dfrac{|x|}{x}$

24. $\lim_{x \to 0} \dfrac{(4 + x)^2 - 16}{x}$

In Exercises 25–34, explore the limit graphically. Confirm algebraically.

25. $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$

26. $\lim_{t \to 2} \dfrac{t^2 - 3t + 2}{t^2 - 4}$

27. $\lim_{x \to 0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2}$

28. $\lim_{x \to 0} \dfrac{\frac{1}{2 + x} - \frac{1}{2}}{x}$

29. $\lim_{x \to 0} \dfrac{(2 + x)^3 - 8}{x}$

30. $\lim_{x \to 0} \dfrac{\sin 2x}{x}$

31. $\lim_{x \to 0} \dfrac{\sin x}{2x^2 - x}$

32. $\lim_{x \to 0} \dfrac{x + \sin x}{x}$

33. $\lim_{x \to 0} \dfrac{\sin^2 x}{x}$

34. $\lim_{x \to 5} \dfrac{x^3 - 125}{x - 5}$

In Exercises 35 and 36, use a graph to explore whether the limit exists.

35. $\lim_{x \to 1} \dfrac{x^2 - 4}{x - 1}$

36. $\lim_{x \to 2} \dfrac{x + 1}{x^2 - 4}$

In Exercises 37–42, determine the limit.

37. $\lim_{x \to 0^+} \operatorname{int} x$

38. $\lim_{x \to 0^-} \operatorname{int} x$

39. $\lim_{x \to 0.01} \operatorname{int} x$

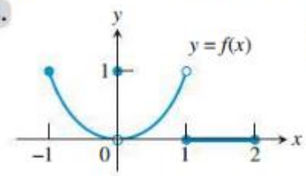
40. $\lim_{x \to 2^-} \operatorname{int} x$

41. $\lim_{x \to 0^+} \dfrac{x}{|x|}$

42. $\lim_{x \to 0^-} \dfrac{x}{|x|}$

In Exercises 43 and 44, which of the statements are true about the function $y = f(x)$ graphed there, and which are false?

43.



(a) $\lim_{x \to -1^+} f(x) = 1$

(b) $\lim_{x \to 0^-} f(x) = 0$

(c) $\lim_{x \to 0^-} f(x) = 1$

(d) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

(e) $\lim_{x \to 0} f(x)$ exists

(f) $\lim_{x \to 0} f(x) = 0$

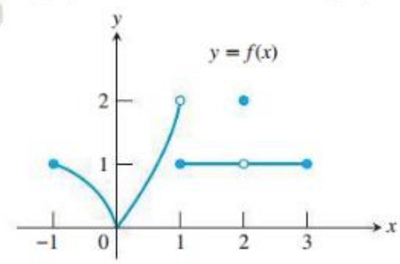
(g) $\lim_{x \to 0} f(x) = 1$

(h) $\lim_{x \to 1} f(x) = 1$

(i) $\lim_{x \to 1} f(x) = 0$

(j) $\lim_{x \to 2^-} f(x) = 2$

44.



(a) $\lim_{x \to -1^+} f(x) = 1$

(b) $\lim_{x \to 2} f(x)$ does not exist.

(c) $\lim_{x \to 2} f(x) = 2$

(d) $\lim_{x \to 1^-} f(x) = 2$

(e) $\lim_{x \to 1^+} f(x) = 1$

(f) $\lim_{x \to 1} f(x)$ does not exist.

(g) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

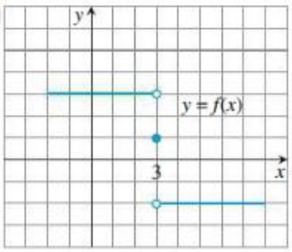
(h) $\lim_{x \to c} f(x)$ exists at every c in $(-1, 1)$.

(i) $\lim_{x \to c} f(x)$ exists at every c in $(1, 3)$.

1.1 EXERCISES # 1, 3, 5, 11, 17, 23, 35, 37, 43, 45, 51, 55, 57, 65

In Exercises 45–50, use the graph to estimate the limits and value of the function, or explain why the limits do not exist.

45.



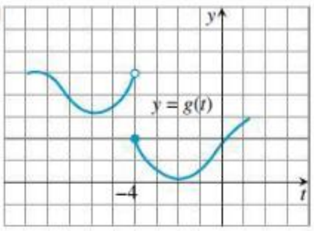
(a) $\lim_{x\to 3^-} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $f(3)$

46.



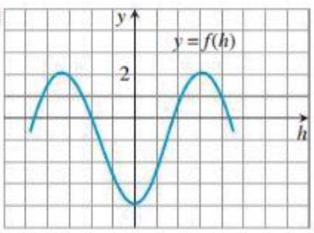
(a) $\lim_{t\to -4^-} g(t)$

(b) $\lim_{t\to -4^+} g(t)$

(c) $\lim_{t\to -4} g(t)$

(d) $g(-4)$

47.



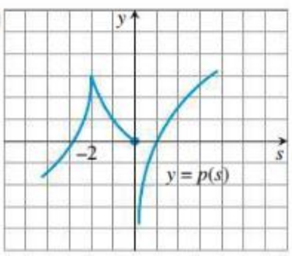
(a) $\lim_{h\to 0^-} f(h)$

(b) $\lim_{h\to 0^+} f(h)$

(c) $\lim_{h\to 0} f(h)$

(d) $f(0)$

48.



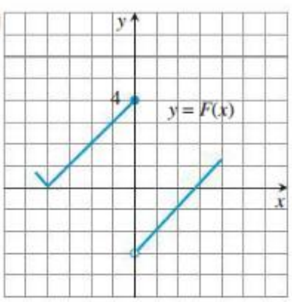
(a) $\lim_{s\to -2^-} p(s)$

(b) $\lim_{s\to -2^+} p(s)$

(c) $\lim_{s\to -2} p(s)$

(d) $p(-2)$

49.



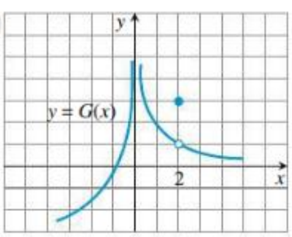
(a) $\lim_{x\to 0^-} F(x)$

(b) $\lim_{x\to 0^+} F(x)$

(c) $\lim_{x\to 0} F(x)$

(d) $F(0)$

50.



(a) $\lim_{x\to 2^-} G(x)$

(b) $\lim_{x\to 2^+} G(x)$

(c) $\lim_{x\to 2} G(x)$

(d) $G(2)$

In Exercises 51–54, match the function with the table.

51. $y_1 = \dfrac{x^2 + x - 2}{x - 1}$ **52.** $y_1 = \dfrac{x^2 - x - 2}{x - 1}$

53. $y_1 = \dfrac{x^2 - 2x + 1}{x - 1}$ **54.** $y_1 = \dfrac{x^2 + x - 2}{x + 1}$

X	Y1
.7	−.4765
.8	−.3111
.9	−.1526
1	0
1.1	.14762
1.2	.29091
1.3	.43043

X = .7

(a)

X	Y1
.7	7.3667
.8	10.8
.9	20.9
1	ERROR
1.1	−18.9
1.2	−8.8
1.3	−5.367

X = .7

(b)

X	Y1
.7	2.7
.8	2.8
.9	2.9
1	ERROR
1.1	3.1
1.2	3.2
1.3	3.3

X = .7

(c)

X	Y1
.7	−.3
.8	−.2
.9	−.1
1	ERROR
1.1	.1
1.2	.2
1.3	.3

X = .7

(d)

In Exercises 55 and 56, determine the limit.

55. Assume that $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = 3$.

(a) $\lim_{x\to 4} (g(x) + 3)$ (b) $\lim_{x\to 4} x f(x)$

(c) $\lim_{x\to 4} g^2(x)$ (d) $\lim_{x\to 4} \dfrac{g(x)}{f(x) - 1}$

56. Assume that $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$.

(a) $\lim_{x\to b} (f(x) + g(x))$ (b) $\lim_{x\to b} (f(x) \cdot g(x))$

(c) $\lim_{x\to b} 4\,g(x)$ (d) $\lim_{x\to b} \dfrac{f(x)}{g(x)}$

In Exercises 57–60, complete parts (a), (b), and (c) for the piecewise-defined function.

(a) Draw the graph of f.

(b) Determine $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$.

(c) **Writing to Learn** Does $\lim_{x\to c} f(x)$ exist? If so, what is it? If not, explain.

57. $c = 2,\ f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

58. $c = 2,\ f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

59. $c = 1,\ f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$

60. $c = -1,\ f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$

1.1 EXERCISES # 1, 3, 5, 11, 17, 23, 35, 37, 43, 45, 51, 55, 57, 65

In Exercises 61–64, complete parts (a)–(d) for the piecewise-defined function.

(a) Draw the graph of f.

(b) At what points c in the domain of f does $\lim_{x\to c} f(x)$ exist?

(c) At what points c does only the left-hand limit exist?

(d) At what points c does only the right-hand limit exist?

61. $f(x) = \begin{cases} \sin x, & -2\pi \le x < 0 \\ \cos x, & 0 \le x \le 2\pi \end{cases}$

62. $f(x) = \begin{cases} \cos x, & -\pi \le x < 0 \\ \sec x, & 0 \le x \le \pi \end{cases}$

63. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

64. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1, \text{ or } x > 1 \end{cases}$

In Exercises 65–68, find the limit graphically. Use the Squeeze Theorem to confirm your answer.

65. $\lim_{x\to 0} x \sin x$

66. $\lim_{x\to 0} x^2 \sin x$

67. $\lim_{x\to 0} x^2 \sin \frac{1}{x^2}$

68. $\lim_{x\to 0} x^2 \cos \frac{1}{x^2}$

69. ***Free Fall*** A water balloon dropped from a window high above the ground falls $y = 4.9t^2$ m in t sec. Find the balloon's

(a) average speed during the first 3 sec of fall.

(b) speed at the instant $t = 3$.

70. ***Free Fall on a Small Airless Planet*** A rock released from rest to fall on a small airless planet falls $y = gt^2$ m in t sec, g a constant. Suppose that the rock falls to the bottom of a crevasse 20 m below and reaches the bottom in 4 sec.

(a) Find the value of g.

(b) Find the average speed for the fall.

(c) With what speed did the rock hit the bottom?

Standardized Test Questions

71. True or False If $\lim_{x\to c^-} f(x) = 2$ and $\lim_{x\to c^+} f(x) = 2$, then $\lim_{x\to c} f(x) = 2$. Justify your answer.

72. True or False $\lim_{x\to 0} \frac{x + \sin x}{x} = 2$. Justify your answer.

In Exercises 73–76, use the following function.

$$f(x) = \begin{cases} 2 - x, & x \le 1 \\ \frac{x}{2} + 1, & x > 1 \end{cases}$$

73. Multiple Choice What is the value of $\lim_{x\to 1^-} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

74. Multiple Choice What is the value of $\lim_{x\to 1^+} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

75. Multiple Choice What is the value of $\lim_{x\to 1} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

76. Multiple Choice What is the value of $f(1)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

77. Group Activity To prove that $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$ when θ is measured in radians, the plan is to show that the right- and left-hand limits are both 1.

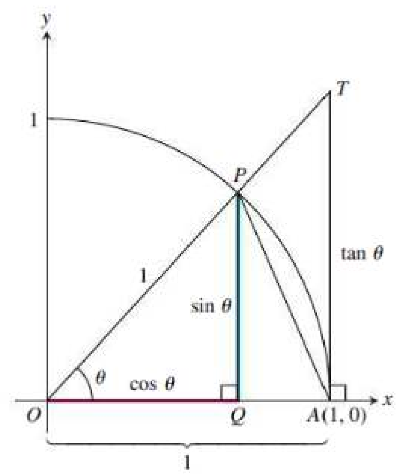
(a) To show that the right-hand limit is 1, explain why we can restrict our attention to $0 < \theta < \pi/2$.

(b) Use the figure to show that

$$\text{area of } \Delta OAP = \frac{1}{2}\sin\theta,$$

$$\text{area of sector } OAP = \frac{\theta}{2},$$

$$\text{area of } \Delta OAT = \frac{1}{2}\tan\theta.$$



(c) Use part (b) and the figure to show that for $0 < \theta < \pi/2$,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

(d) Show that for $0 < \theta < \pi/2$ the inequality of part (c) can be written in the form

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

(e) Show that for $0 < \theta < \pi/2$ the inequality of part (d) can be written in the form

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

(f) Use the Squeeze Theorem to show that

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

1.1 EXERCISES # 1, 3, 5, 11, 17, 23, 35, 37, 43, 45, 51, 55, 57, 65

(g) Show that $(\sin\theta)/\theta$ is an even function.

(h) Use part (g) to show that

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta}=1.$$

(i) Finally, show that

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=1.$$

Extending the Ideas

78. *Controlling Outputs* Let $f(x)=\sqrt{3x-2}$.

(a) Show that $\lim_{x\to 2} f(x)=2=f(2)$.

(b) Use a graph to estimate values for a and b so that $1.8<f(x)<2.2$ provided $a<x<b$.

(c) Use a graph to estimate values for a and b so that $1.99<f(x)<2.01$ provided $a<x<b$.

79. *Controlling Outputs* Let $f(x)=\sin x$.

(a) Find $f(\pi/6)$.

(b) Use a graph to estimate an interval (a, b) about $x=\pi/6$ so that $0.3<f(x)<0.7$ provided $a<x<b$.

(c) Use a graph to estimate an interval (a, b) about $x=\pi/6$ so that $0.49<f(x)<0.51$ provided $a<x<b$.

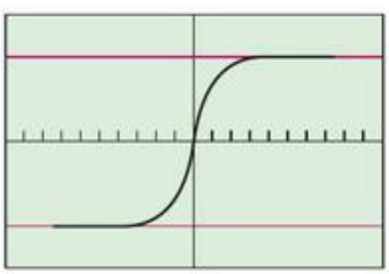
80. *Limits and Geometry* Let $P(a, a^2)$ be a point on the parabola $y=x^2$, $a>0$. Let O be the origin and $(0, b)$ the y-intercept of the perpendicular bisector of line segment OP. Find $\lim_{P\to O} b$.

1.2 Limits Involving Infinity

1.2 Limits Involving Infinity

You will be able to interpret, estimate, and determine infinite limits and limits at infinity.

- The Squeeze Theorem for limits at infinity
- Asymptotic and unbounded behavior of functions
- End behavior of functions



[−10, 10] by [−1.5, 1.5]

Finite Limits as $x\to\pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, when we say, "The limit of f as x approaches infinity is L," we mean that we can force the value of the function to be as close as we wish to L by taking values of x that are sufficiently far to the right on the number line. When we say, "The limit of f as x approaches negative infinity $(-\infty)$ is L," we mean that we can force the value of the function to be as close as we wish to L by taking values of x that are sufficiently far to the left. (The limit in each case may or may not exist.)

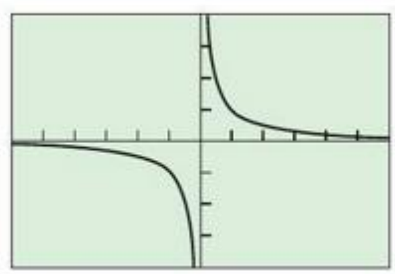
Looking at $f(x)=1/x$ (Figure 1.9), we observe

(a) as $x\to\infty$, $(1/x)\to 0$ and we write

$$\lim_{x\to\infty}(1/x)=0,$$

(b) as $x\to-\infty$, $(1/x)\to 0$ and we write

$$\lim_{x\to-\infty}(1/x)=0.$$



[−6, 6] by [−4, 4]

Figure 1.9 The graph of $f(x)=1/x$.

X	Y_1	
0	0	
1	.7071	
2	.8944	
3	.9487	
4	.9701	
5	.9806	
6	.9864	
$Y_1 = X/\sqrt{(X^2+1)}$		

X	Y_1	
-6	-.9864	
-5	-.9806	
-4	-.9701	
-3	-.9487	
-2	-.8944	
-1	-.7071	
0	0	
$Y_1 = X/\sqrt{(X^2+1)}$		

(b)

Figure 1.10 (a) The graph of $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes, $y = -1$ and $y = 1$. (b) Selected values of f. (Example 1)

We say that the line $y = 0$ is a *horizontal asymptote* of the graph of f.

Definition Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

The graph of $f(x) = 2 + (1/x)$ has the single horizontal asymptote $y = 2$ because

$$\lim_{x\to\infty}\left(2+\frac{1}{x}\right) = 2 \quad \text{and} \quad \lim_{x\to-\infty}\left(2+\frac{1}{x}\right) = 2.$$

A function can have more than one horizontal asymptote, as Example 1 demonstrates.

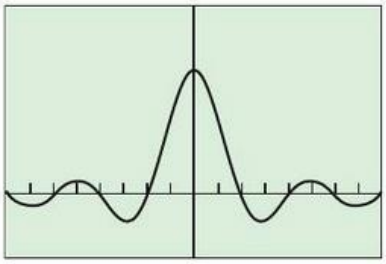
Example 1 Looking for Horizontal Asymptotes

Use graphs and tables to find $\lim_{x\to\infty} f(x)$, to find $\lim_{x\to-\infty} f(x)$, and to identify all horizontal asymptotes of $f(x) = x/\sqrt{x^2+1}$.

Solution

Figure 1.10a shows the graph for $-10 \le x \le 10$. The graph climbs rapidly toward the line $y = 1$ as x moves away from the origin to the right. On our calculator screen, the graph soon becomes indistinguishable from the line. Thus $\lim_{x\to\infty} f(x) = 1$. Similarly, as x moves away from the origin to the left, the graph drops rapidly toward the line $y = -1$ and soon appears to overlap the line. Thus $\lim_{x\to-\infty} f(x) = -1$. The horizontal asymptotes are $y = 1$ and $y = -1$.

continued



$[-4\pi, 4\pi]$ by $[-0.5, 1.5]$

(a)

X	Y_1
100	−.0051
200	−.0044
300	−.0033
400	−.0021
500	−9E−4
600	7.4E−5
700	7.8E−4

Y_1 = sin(X)/X

(b)

Figure 1.11 (a) The graph of $f(x) = (\sin x)/x$ oscillates about the x-axis. The amplitude of the oscillations decreases toward zero as $x \to \pm\infty$. (b) A table of values for f that suggests $f(x) \to 0$ as $x \to \infty$. (Example 2)

The table in Figure 1.10b confirms the rapid approach of $f(x)$ toward 1 as $x \to \infty$. Since f is an odd function of x, we can expect its values to approach -1 in a similar way as $x \to -\infty$. ***Now Try Exercise 5.***

Squeeze Theorem Revisited

The Squeeze Theorem also holds for limits as $x \to \pm\infty$.

Example 2 Finding a Limit as x Approaches ∞

Find $\lim_{x\to\infty} f(x)$ for $f(x) = \dfrac{\sin x}{x}$.

Solution

The graph and table of values in Figure 1.11 suggest that $y = 0$ is the horizontal asymptote of f.

We know that $-1 \le \sin x \le 1$. So for $x > 0$ we have

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Therefore, by the Squeeze Theorem,

$$0 = \lim_{x\to\infty}\left(-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\sin x}{x} = \lim_{x\to\infty}\frac{1}{x} = 0.$$

Since $(\sin x)/x$ is an even function of x, we can also conclude that

$$\lim_{x\to-\infty}\frac{\sin x}{x} = 0.$$

Now Try Exercise 9.

Limits at infinity have properties similar to those of finite limits.

Notational Fluency Note

$\lim_{x\to\infty} f(x)$ has the same meaning as $\lim_{x\to+\infty} f(x)$. For a limit that applies to both extremes, use $\lim_{x\to\pm\infty} f(x)$.

Theorem 5 Properties of Limits as $x \to \pm\infty$

If L, M, and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad\text{and}\quad \lim_{x\to\pm\infty} g(x) = M, \quad\text{then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty}(f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty}(f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty}(k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$
6. *Power Rule:* If r and s are integers, $s \neq 0$, then

$$\lim_{x\to\pm\infty}(f(x))^{r/s} = L^{r/s},$$

provided that $L^{r/s}$ is a real number and $L > 0$ if s is even.

We can use Theorem 5 to find limits at infinity of functions with complicated expressions, as illustrated in Example 3.

Example 3 Using Theorem 5

Find $\lim_{x \to \infty} \frac{5x + \sin x}{x}$.

Solution

Notice that

$$\frac{5x + \sin x}{x} = \frac{5x}{x} + \frac{\sin x}{x} = 5 + \frac{\sin x}{x}.$$

Therefore,

$$\lim_{x \to \infty} \frac{5x + \sin x}{x} = \lim_{x \to \infty} 5 + \lim_{x \to \infty} \frac{\sin x}{x} \quad \text{Sum Rule}$$

$$= 5 + 0 = 5. \quad \text{Known values}$$

Now Try Exercise 25.

Exploration 1 Exploring Theorem 5

We must be careful how we apply Theorem 5.

1. (Example 3 again) Let $f(x) = 5x + \sin x$ and $g(x) = x$. Do the limits as $x \to \infty$ of f and g exist? Can we apply the Quotient Rule to $\lim_{x \to \infty} f(x)/g(x)$? Explain. Does the limit of the quotient exist?
2. Let $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$. Describe the behavior of f and g as $x \to \infty$. Can we apply the Sum Rule to $\lim_{x \to \infty} (f(x) + g(x))$? Explain. Does the limit of the sum exist?
3. Let $f(x) = \ln(2x)$ and $g(x) = \ln(x + 1)$. Find the limits as $x \to \infty$ of f and g. Can we apply the Difference Rule to $\lim_{x \to \infty} (f(x) - g(x))$? Explain. Does the limit of the difference exist?
4. Based on parts 1–3, what advice might you give about applying Theorem 5?

Infinite Limits as $x \to a$

If the values of a function $f(x)$ outgrow all positive bounds as x approaches a finite number a, we say that $\lim_{x \to a} f(x) = \infty$. If the values of f become large and negative, exceeding all negative bounds as $x \to a$, we say that $\lim_{x \to a} f(x) = -\infty$.

Looking at $f(x) = 1/x$ (Figure 1.9, page 74), we observe that

$$\lim_{x \to 0^+} 1/x = \infty \quad \text{and} \quad \lim_{x \to 0^-} 1/x = -\infty.$$

We say that the line $x = 0$ is a *vertical asymptote* of the graph of f.

Definition Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty.$$

Example 4 Finding Vertical Asymptotes

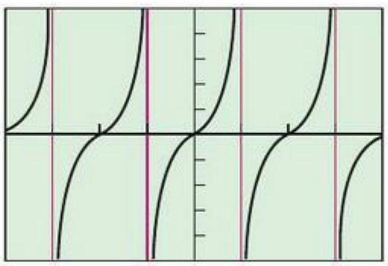
Find the vertical asymptotes of $f(x) = \frac{1}{x^2}$. Describe the behavior to the left and right of each vertical asymptote.

Solution

The values of the function approach ∞ on either side of $x = 0$.

$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x^2} = \infty.$$

The line $x = 0$ is the only vertical asymptote. ***Now Try Exercise 27.***



$[-2\pi, 2\pi]$ by $[-5, 5]$

Figure 1.12 The graph of $f(x) = \tan x$ has a vertical asymptote at $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. (Example 5)

We can also say that $\lim_{x \to 0} (1/x^2) = \infty$. We can make no such statement about $1/x$.

Example 5 Finding Vertical Asymptotes

The graph of $f(x) = \tan x = (\sin x)/(\cos x)$ has infinitely many vertical asymptotes, one at each point where the cosine is zero. If a is an odd multiple of $\pi/2$, then

$$\lim_{x \to a^+} \tan x = -\infty \quad \text{and} \quad \lim_{x \to a^-} \tan x = \infty,$$

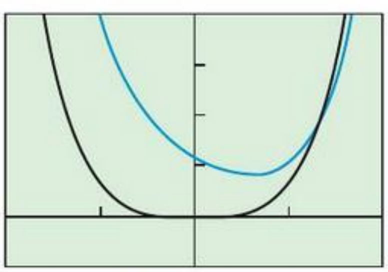
as suggested by Figure 1.12. ***Now Try Exercise 31.***

You might think that the graph of a quotient always has a vertical asymptote where the denominator is zero, but that need not be the case. For example, we observed in Section 2.1 that $\lim_{x \to 0} (\sin x)/x = 1$.

End Behavior Models

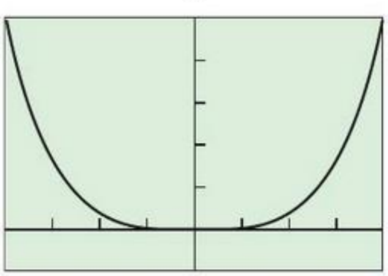
For numerically large values of x, we can sometimes model the behavior of a complicated function by a simpler one that acts virtually in the same way.

$y = 3x^4 - 2x^3 + 3x^2 - 5x + 6$



$[-2, 2]$ by $[-5, 20]$

(a)



$[-20, 20]$ by $[-100000, 500000]$

(b)

Figure 1.13 The graphs of f and g, (a) distinct for $|x|$ small, are (b) nearly identical for $|x|$ large. (Example 6)

Example 6 Modeling Functions for $|x|$ Large

Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although f and g are quite different for numerically small values of x, they are virtually identical for $|x|$ large.

Solution

The graphs of f and g (Figure 1.13a), quite different near the origin, are virtually identical on a larger scale (Figure 1.13b).

We can test the claim that g models f for numerically large values of x by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\begin{aligned} \lim_{x \to \pm\infty} \frac{f(x)}{g(x)} &= \lim_{x \to \pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x \to \pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right) \\ &= 1, \end{aligned}$$

convincing evidence that f and g behave alike for $|x|$ large. ***Now Try Exercise 39.***

Definition End Behavior Model

The function g is

(a) a **right end behavior model** for f if and only if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

(b) a **left end behavior model** for f if and only if $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = 1$.

If one function provides both a left and a right end behavior model, it is simply called an **end behavior model.** Thus, $g(x) = 3x^4$ is an end behavior model for $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ (Example 6).

In general, $g(x) = a_n x^n$ is an end behavior model for the polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, a_n \neq 0$. Overall, the end behavior of all polynomials is like the end behavior of monomials. This is the key to the end behavior of rational functions, as illustrated in Example 7.

Example 7 Finding End Behavior Models

Find an end behavior model for

(a) $f(x) = \dfrac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$ **(b)** $g(x) = \dfrac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5}$

Solution

(a) Notice that $2x^5$ is an end behavior model for the numerator of f, and $3x^2$ is an end behavior model for the denominator. This makes

$$\frac{2x^5}{3x^2} = \frac{2}{3}x^3$$

an end behavior model for f.

(b) Similarly, $2x^3$ is an end behavior model for the numerator of g, and $5x^3$ is an end behavior model for the denominator of g. This makes

$$\frac{2x^3}{5x^3} = \frac{2}{5}$$

an end behavior model for g. ***Now Try Exercise 43.***

Notice in Example 7b that the end behavior model for g, $y = 2/5$, is also a horizontal asymptote of the graph of g, while in 7a, the graph of f does not have a horizontal asymptote. We can use the end behavior model of a rational function to identify any horizontal asymptote.

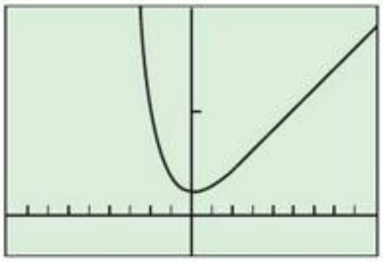
We can see from Example 7 that a rational function always has a simple power function as an end behavior model.

A function's right and left end behavior models need not be the same function.

Example 8 Finding End Behavior Models

Let $f(x) = x + e^{-x}$. Show that $g(x) = x$ is a right end behavior model for f, while $h(x) = e^{-x}$ is a left end behavior model for f.

continued



[–9, 9] by [–2, 10]

Figure 1.14 The graph of $f(x) = x + e^{-x}$ looks like the graph of $g(x) = x$ to the right of the y-axis, and like the graph of $h(x) = e^{-x}$ to the left of the y-axis. (Example 8)

Solution

On the right,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x + e^{-x}}{x} = \lim_{x\to\infty}\left(1 + \frac{e^{-x}}{x}\right) = 1 \text{ because } \lim_{x\to\infty} \frac{e^{-x}}{x} = 0.$$

On the left,

$$\lim_{x\to-\infty} \frac{f(x)}{h(x)} = \lim_{x\to-\infty} \frac{x + e^{-x}}{e^{-x}} = \lim_{x\to-\infty}\left(\frac{x}{e^{-x}} + 1\right) = 1 \text{ because } \lim_{x\to-\infty} \frac{x}{e^{-x}} = 0.$$

The graph of f in Figure 1.14 supports these end behavior conclusions.

Now Try Exercise 45.

"Seeing" Limits as $x \to \pm\infty$

We can investigate the graph of $y = f(x)$ as $x \to \pm\infty$ by investigating the graph of $y = f(1/x)$ as $x \to 0$.

Example 9 Using Substitution

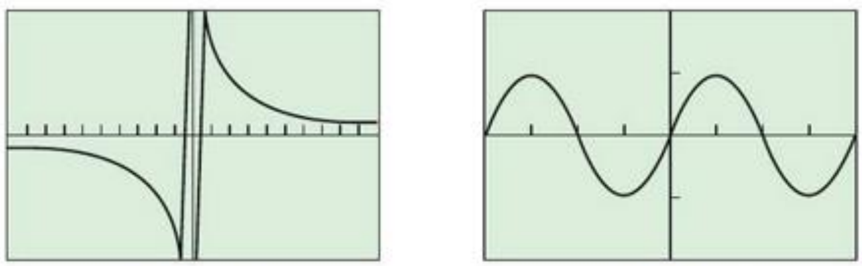
Find $\lim_{x\to\infty} \sin(1/x)$.

Solution

Figure 1.15a suggests that the limit is 0. Indeed, replacing $\lim_{x\to\infty} \sin(1/x)$ by the equivalent $\lim_{x\to 0^+} \sin x = 0$ (Figure 1.15b), we find

$$\lim_{x\to\infty} \sin(1/x) = \lim_{x\to 0^+} \sin x = 0.$$

Now Try Exercise 49.



[–10, 10] by [–1, 1]

(a)

[–2π, 2π] by [–2, 2]

(b)

Figure 1.15 The graphs of (a) $f(x) = \sin(1/x)$ and (b) $g(x) = f(1/x) = \sin x$. (Example 9)

QUICK REVIEW All # 1 - 10

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, find f^{-1} and graph f, f^{-1}, and $y = x$ in the same square viewing window.

1. $f(x) = 2x - 3$

2. $f(x) = e^x$

3. $f(x) = \tan^{-1} x$

4. $f(x) = \cot^{-1} x$

In Exercises 5 and 6, find the quotient $q(x)$ and remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

5. $f(x) = 2x^3 - 3x^2 + x - 1, \quad g(x) = 3x^3 + 4x - 5$

6. $f(x) = 2x^5 - x^3 + x - 1, \quad g(x) = x^3 - x^2 + 1$

In Exercises 7–10, write a formula for **(a)** $f(-x)$ and **(b)** $f(1/x)$. Simplify where possible.

7. $f(x) = \cos x$

8. $f(x) = e^{-x}$

9. $f(x) = \dfrac{\ln|x|}{x}$

10. $f(x) = \left(x + \dfrac{1}{x}\right)\sin x$

1.2 EXERCISES # 1, 9, 13, 21, 27, 35, 41, 45, 49, 54

In Exercises 1–8, use graphs and tables to find **(a)** $\lim_{x\to\infty} f(x)$ and **(b)** $\lim_{x\to-\infty} f(x)$. **(c)** Identify all horizontal asymptotes.

1. $f(x) = \cos\left(\frac{1}{x}\right)$

2. $f(x) = \frac{\sin 2x}{x}$

3. $f(x) = \frac{e^{-x}}{x}$

4. $f(x) = \frac{3x^3 - x + 1}{x + 3}$

5. $f(x) = \frac{3x + 1}{|x| + 2}$

6. $f(x) = \frac{2x - 1}{|x| - 3}$

7. $f(x) = \frac{x}{|x|}$

8. $f(x) = \frac{|x|}{|x| + 1}$

In Exercises 9–12, find the limit and confirm your answer using the Squeeze Theorem.

9. $\lim_{x\to\infty} \frac{1 - \cos x}{x^2}$

10. $\lim_{x\to-\infty} \frac{1 - \cos x}{x^2}$

11. $\lim_{x\to-\infty} \frac{\sin x}{x}$

12. $\lim_{x\to\infty} \frac{\sin(x^2)}{x}$

In Exercises 13–20, use graphs and tables to find the limits.

13. $\lim_{x\to2^+} \frac{1}{x - 2}$

14. $\lim_{x\to2^-} \frac{x}{x - 2}$

15. $\lim_{x\to-3^-} \frac{1}{x + 3}$

16. $\lim_{x\to-3^+} \frac{x}{x + 3}$

17. $\lim_{x\to0^+} \frac{\text{int } x}{x}$

18. $\lim_{x\to0^-} \frac{\text{int } x}{x}$

19. $\lim_{x\to0^+} \csc x$

20. $\lim_{x\to(\pi/2)^+} \sec x$

In Exercises 21–26, find $\lim_{x\to\infty} y$ and $\lim_{x\to-\infty} y$.

21. $y = \left(2 - \frac{x}{x + 1}\right)\left(\frac{x^2}{5 + x^2}\right)$

22. $y = \left(\frac{2}{x} + 1\right)\left(\frac{5x^2 - 1}{x^2}\right)$

23. $y = \frac{\cos(1/x)}{1 + (1/x)}$

24. $y = \frac{2x + \sin x}{x}$

25. $y = \frac{\cos x - 2x^3}{x^3}$

26. $y = \frac{x \sin x + 2\cos x}{2x^2}$

In Exercises 27–34, **(a)** find the vertical asymptotes of the graph of $f(x)$. **(b)** Describe the behavior of $f(x)$ to the left and right of each vertical asymptote.

27. $f(x) = \frac{1}{x^2 - 4}$

28. $f(x) = \frac{x^2 - 1}{2x + 4}$

29. $f(x) = \frac{x^2 - 2x}{x + 1}$

30. $f(x) = \frac{1 - x}{2x^2 - 5x - 3}$

31. $f(x) = \cot x$

32. $f(x) = \sec x$

33. $f(x) = \frac{\tan x}{\sin x}$

34. $f(x) = \frac{\cot x}{\cos x}$

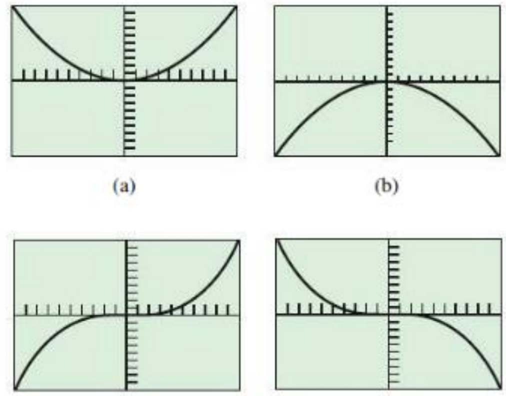
In Exercises 35–38, match the function with the graph of its end behavior model.

35. $y = \frac{2x^3 - 3x^2 + 1}{x + 3}$

36. $y = \frac{x^5 - x^4 + x + 1}{2x^2 + x - 3}$

37. $y = \frac{2x^4 - x^3 + x^2 - 1}{2 - x}$

38. $y = \frac{x^4 - 3x^3 + x^2 - 1}{1 - x^2}$



(a) (b) (c) (d)

In Exercises 39–44, **(a)** find a power function end behavior model for f. **(b)** Identify any horizontal asymptotes.

39. $f(x) = 3x^2 - 2x + 1$

40. $f(x) = -4x^3 + x^2 - 2x - 1$

41. $f(x) = \frac{x - 2}{2x^2 + 3x - 5}$

42. $f(x) = \frac{3x^2 - x + 5}{x^2 - 4}$

43. $f(x) = \frac{4x^3 - 2x + 1}{x - 2}$

44. $f(x) = \frac{-x^4 + 2x^2 + x - 3}{x^2 - 4}$

In Exercises 45–48, find **(a)** a simple basic function as a right end behavior model and **(b)** a simple basic function as a left end behavior model for the function.

45. $y = e^x - 2x$

46. $y = x^2 + e^{-x}$

47. $y = x + \ln|x|$

48. $y = x^2 + \sin x$

In Exercises 49–52, use the graph of $y = f(1/x)$ to find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

49. $f(x) = xe^x$

50. $f(x) = x^2 e^{-x}$

51. $f(x) = \frac{\ln|x|}{x}$

52. $f(x) = x \sin\frac{1}{x}$

In Exercises 53 and 54, find the limit of $f(x)$ as **(a)** $x \to -\infty$, **(b)** $x \to \infty$, **(c)** $x \to 0^-$, and **(d)** $x \to 0^+$.

53. $f(x) = \begin{cases} 1/x, & x < 0 \\ -1, & x \geq 0 \end{cases}$

54. $f(x) = \begin{cases} \frac{x - 2}{x - 1}, & x \leq 0 \\ 1/x^2, & x > 0 \end{cases}$

Group Activity In Exercises 55 and 56, sketch a graph of a function $y = f(x)$ that satisfies the stated conditions. Include any asymptotes.

55. $\lim_{x\to1} f(x) = 2$, $\lim_{x\to5^-} f(x) = \infty$, $\lim_{x\to5^+} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -1$, $\lim_{x\to-2^+} f(x) = -\infty$, $\lim_{x\to-2^-} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$

56. $\lim_{x\to2} f(x) = -1$, $\lim_{x\to4^+} f(x) = -\infty$, $\lim_{x\to4^-} f(x) = \infty$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 2$

1.2 EXERCISES # 1, 9, 13, 21, 27, 35, 41, 45, 49, 54

57. Group Activity *End Behavior Models* Suppose that $g_1(x)$ is a right end behavior model for $f_1(x)$ and that $g_2(x)$ is a right end behavior model for $f_2(x)$. Explain why this makes $g_1(x)/g_2(x)$ a right end behavior model for $f_1(x)/f_2(x)$.

58. Writing to Learn Let L be a real number, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = \infty$ or $-\infty$. Can $\lim_{x\to c} (f(x) + g(x))$ be determined? Explain.

Standardized Test Questions

59. True or False It is possible for a function to have more than one horizontal asymptote. Justify your answer.

60. True or False If $f(x)$ has a vertical asymptote at $x = c$, then either $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \infty$ or $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = -\infty$. Justify your answer.

61. Multiple Choice $\lim_{x\to 2^-} \frac{x}{x-2} =$

(A) $-\infty$ **(B)** ∞ **(C)** 1 **(D)** $-1/2$ **(E)** -1

You may use a graphing calculator to solve the following problems.

62. Multiple Choice $\lim_{x\to 0} \frac{\cos(2x)}{x} =$

(A) $1/2$ **(B)** 1 **(C)** 2 **(D)** $\cos 2$ **(E)** does not exist

63. Multiple Choice $\lim_{x\to 0} \frac{\sin(3x)}{x} =$

(A) $1/3$ **(B)** 1 **(C)** 3 **(D)** $\sin 3$ **(E)** does not exist

64. Multiple Choice Which of the following is an end behavior for

$$f(x) = \frac{2x^3 - x^2 + x + 1}{x^3 - 1}?$$

(A) x^3 **(B)** $2x^3$ **(C)** $1/x^3$ **(D)** 2 **(E)** $1/2$

Exploration

65. *Exploring Properties of Limits* Find the limits of f, g, and fg as $x \to c$.

(a) $f(x) = \frac{1}{x}, \quad g(x) = x, \quad c = 0$

(b) $f(x) = -\frac{2}{x^3}, \quad g(x) = 4x^3, \quad c = 0$

(c) $f(x) = \frac{3}{x-2}, \quad g(x) = (x-2)^3, \quad c = 2$

(d) $f(x) = \frac{5}{(3-x)^4}, \quad g(x) = (x-3)^2, \quad c = 3$

(e) Writing to Learn Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \infty$. Based on your observations in parts (a)–(d), what can you say about $\lim_{x\to c} (f(x) \cdot g(x))$?

Extending the Ideas

66. *The Greatest Integer Function*

(a) Show that

$$\frac{x-1}{x} < \frac{\text{int } x}{x} \le 1 \ (x > 0) \text{ and } \frac{x-1}{x} > \frac{\text{int } x}{x} \ge 1 \ (x < 0).$$

(b) Determine $\lim_{x\to\infty} \frac{\text{int } x}{x}$.

(c) Determine $\lim_{x\to-\infty} \frac{\text{int } x}{x}$.

67. *Squeeze Theorem* Use the Squeeze Theorem to confirm the limit as $x \to \infty$ found in Exercise 3.

68. Writing to Learn Explain why there is no value L for which $\lim_{x\to\infty} \sin x = L$.

In Exercises 69–71, find the limit. Give a convincing argument that the value is correct.

69. $\lim_{x\to\infty} \frac{\ln x^2}{\ln x}$

70. $\lim_{x\to\infty} \frac{\ln x}{\log x}$

71. $\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x}$

1.3 Continuity

1.3 Continuity

You will be able to analyze functions to find intervals of continuity and points of discontinuity and to determine the applicability of the Intermediate Value Theorem.

- Definition of continuity at a point
- Types of discontinuities
- Sums, differences, products, quotients, and compositions of continuous functions
- Common continuous functions
- Continuity and the Intermediate Value Theorem

Continuity at a Point

When we plot function values generated in the laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure (Figure 1.16). In doing so, we are assuming that we are working with a *continuous function,* a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. Any function $y = f(x)$ whose graph can be sketched in one continuous motion without lifting the pencil is an example of a continuous function.

Continuous functions are those we use to find a planet's closest point of approach to the sun or the peak concentration of antibodies in blood plasma. They are also the functions we use to describe how a body moves through space or how the speed of a chemical reaction changes with time. In fact, so many physical processes proceed continuously that throughout the 18th and 19th centuries it rarely occurred to anyone to look for any other kind of behavior. It came as a surprise when the physicists of the 1920s discovered that light comes in particles and that heated atoms emit light at discrete frequencies (Figure 1.17). As a result of these and other discoveries, and because of the heavy use of discontinuous functions in computer science, statistics, and mathematical modeling, the issue of continuity has become one of practical as well as theoretical importance.

To understand continuity, we need to consider a function like the one in Figure 1.18, whose limits we investigated in Example 8, Section 2.1.

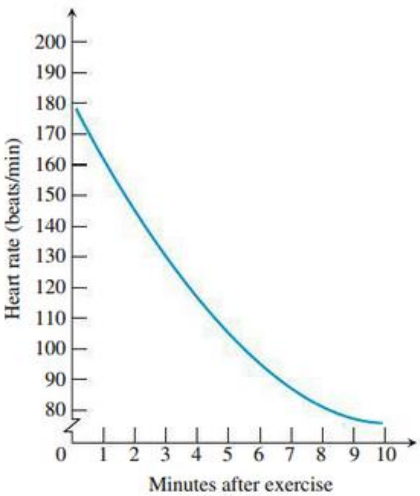


Figure 1.16 How the heartbeat returns to a normal rate after running.

Figure 1.17 The laser was developed as a result of an understanding of the nature of the atom.

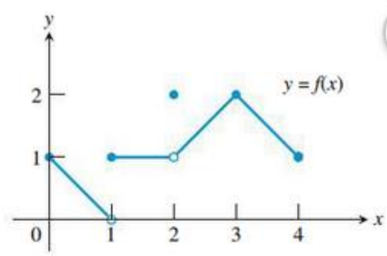


Figure 1.18 The function is continuous on $[0, 4]$ except at $x = 1$ and $x = 2$. (Example 1)

Example 1 Investigating Continuity

Find the points at which the function f in Figure 1.18 is continuous, and the points at which f is discontinuous.

Solution

The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1$ and $x = 2$. At these points there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

Points at which f is continuous:

At $x = 0$, $\quad \lim_{x \to 0^+} f(x) = f(0)$.

At $x = 4$, $\quad \lim_{x \to 4^-} f(x) = f(4)$.

At $0 < c < 4$, $c \neq 1, 2$, $\quad \lim_{x \to c} f(x) = f(c)$.

continued

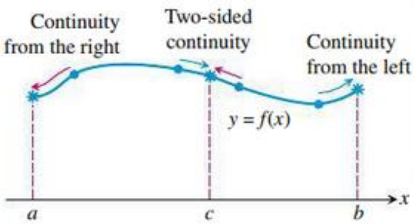


Figure 1.19 Continuity at points a, b, and c for a function $y = f(x)$ that is continuous on the interval $[a, b]$.

Points at which f is discontinuous:

At $x = 1$, $\lim_{x\to 1} f(x)$ does not exist.

At $x = 2$, $\lim_{x\to 2} f(x) = 1$, but $1 \neq f(2)$.

At $c < 0, c > 4$, these points are not in the domain of f. **Now Try Exercise 5.**

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit) (Figure 1.19).

Definition Continuity at a Point

Interior Point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x\to a^+} f(x) = f(a) \text{ or } \lim_{x\to b^-} f(x) = f(b), \text{ respectively.}$$

The assumption that exponential functions such as 2^x are continuous makes it possible to define a number raised to an irrational exponent such as 2^π. Fractional exponents come from combining integer roots and powers: $2^{2/3} = \left(\sqrt[3]{2}\right)^2$. This means that $2^{3.14} = 2^{314/100}$ is the 314th power of the 100th root of 2. As we add more decimal digits, we get closer to the value of 2^π, which is *defined* to be the limit of 2^x as x approaches π through rational numbers.

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and that c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

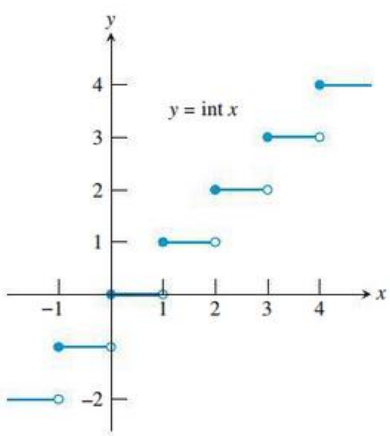


Figure 1.20 The function int x is continuous at every noninteger point. (Example 2)

Example 2 Finding Points of Continuity and Discontinuity

Find the points of continuity and the points of discontinuity of the greatest integer function (Figure 1.20).

Solution

For the function to be continuous at $x = c$, the limit as $x \to c$ must exist and must equal the value of the function at $x = c$. The greatest integer function is discontinuous at every integer. For example,

$$\lim_{x\to 3^-} \text{int } x = 2 \quad \text{and} \quad \lim_{x\to 3^+} \text{int } x = 3,$$

so the limit as $x \to 3$ does not exist. Notice that int $3 = 3$. In general, if n is any integer,

$$\lim_{x\to n^-} \text{int } x = n - 1 \quad \text{and} \quad \lim_{x\to n^+} \text{int } x = n,$$

so the limit as $x \to n$ does not exist.

The greatest integer function is continuous at every other real number. For example,

$$\lim_{x\to 1.5} \text{int } x = 1 = \text{int } 1.5.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x\to c} \text{int } x = n - 1 = \text{int } c.$$

Now Try Exercise 7.

Shirley Ann Jackson (1946–)

Distinguished scientist Shirley Jackson credits her interest in science to her parents and excellent mathematics and science teachers in high school. She studied physics and, in 1973, became the first African American woman to earn a Ph.D. at the Massachusetts Institute of Technology. Since then, Dr. Jackson has done research on topics related to theoretical material sciences, has received numerous scholarships and honors, and has published more than 100 scientific articles.

Figure 1.21 is a catalog of discontinuity types. The function in (a) is continuous at $x = 0$. The function in (b) would be continuous if it had $f(0) = 1$. The function in (c) would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in (b) and (c) are **removable.** Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in (d)–(f) of Figure 1.21 are more serious: $\lim_{x\to 0} f(x)$ does not exist and there is no way to improve the situation by changing f at 0. The step function in (d) has a **jump discontinuity:** The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in (e) has an **infinite discontinuity.** The function in (f) has an **oscillating discontinuity:** It oscillates all the way from -1 to $+1$ no matter how close to 0 we take x.

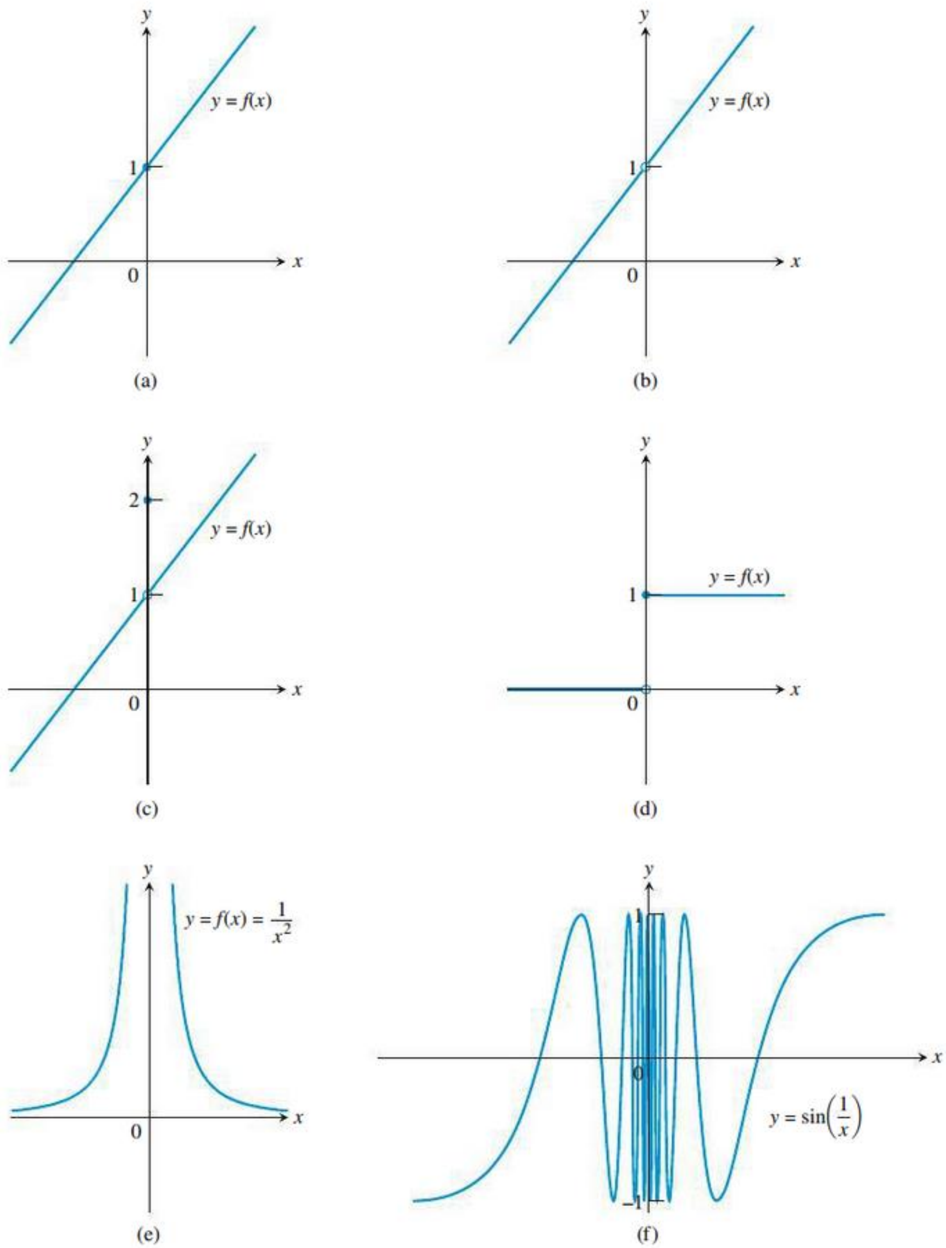


Figure 1.21 The function in part (a) is continuous at $x = 0$. The functions in parts (b)–(f) are not.

Exploration 1 Removing a Discontinuity

Let $f(x) = \dfrac{x^3 - 7x - 6}{x^2 - 9}$.

1. Factor the denominator. What is the domain of f?
2. Investigate the graph of f around $x = 3$ to see that f has a removable discontinuity at $x = 3$.
3. How should f be defined at $x = 3$ to remove the discontinuity? Use ZOOM-IN and tables as necessary.
4. Show that $(x - 3)$ is a factor of the numerator of f, and remove all common factors. Now compute the limit as $x \to 3$ of the reduced form for f.
5. Show that the *extended function*

$$g(x) = \begin{cases} \dfrac{x^3 - 7x - 6}{x^2 - 9}, & x \neq 3 \\ 10/3, & x = 3 \end{cases}$$

is continuous at $x = 3$. The function g is the **continuous extension** of the original function f to include $x = 3$. *Now Try Exercise 25.*

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$.

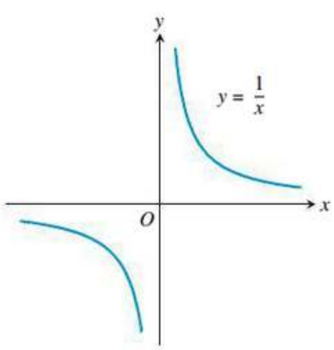


Figure 1.22 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$. (Example 3)

Example 3 Identifying Continuous Functions

The reciprocal function $y = 1/x$ (Figure 1.22) is a continuous function because it is continuous at every point of its domain. However, it has a point of discontinuity at $x = 0$ because it is not defined there. *Now Try Exercise 31.*

Polynomial functions f are continuous at every real number c because $\lim_{x \to c} f(x) = f(c)$. Rational functions are continuous at every point of their domains. They have points of discontinuity at the zeros of their denominators. The absolute value function $y = |x|$ is continuous at every real number. The exponential functions, logarithmic functions, trigonometric functions, and radical functions like $y = \sqrt[n]{x}$ (n a positive integer greater than 1) are continuous at every point of their domains. All of these functions are continuous functions.

Algebraic Combinations

As you may have guessed, algebraic combinations of continuous functions are continuous wherever they are defined.

Theorem 6 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Products:* $f \cdot g$
4. *Constant multiples:* $k \cdot f$, for any number k
5. *Quotients:* f/g, provided $g(c) \neq 0$

Composites

All composites of continuous functions are continuous. This means composites like

$$y = \sin(x^2) \quad \text{and} \quad y = |\cos x|$$

are continuous at every point at which they are defined. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 1.23). In this case, the limit as $x \to c$ is $g(f(c))$.

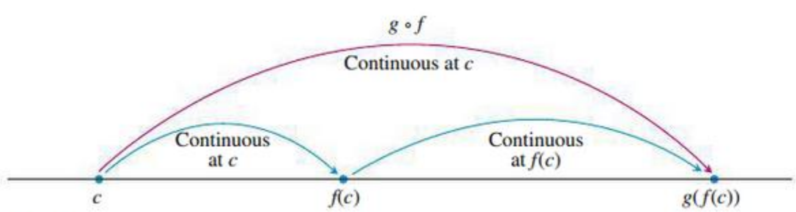


Figure 1.23 Composites of continuous functions are continuous.

Theorem 7 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

Example 4 Using Theorem 7

Show that $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ is continuous.

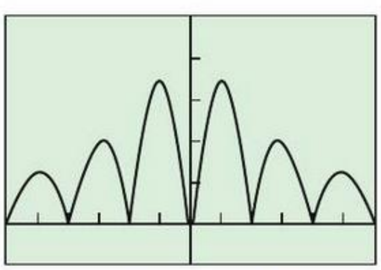
Solution

The graph (Figure 1.24) of $y = |(x \sin x)/(x^2 + 2)|$ suggests that the function is continuous at every value of x. By letting

$$g(x) = |x| \quad \text{and} \quad f(x) = \frac{x \sin x}{x^2 + 2},$$

we see that y is the composite $g \circ f$.

We know that the absolute value function g is continuous. The function f is continuous by Theorem 6. Their composite is continuous by Theorem 7. ***Now Try Exercise 33.***



$[-3\pi, 3\pi]$ by $[-0.1, 0.5]$

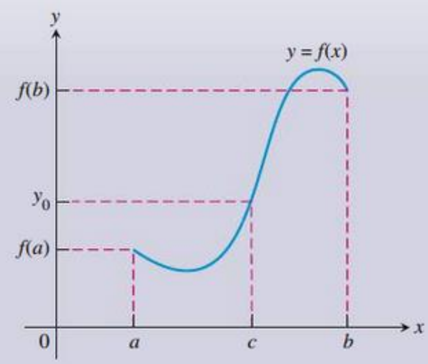
Figure 1.24 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous. (Example 4)

Intermediate Value Theorem for Continuous Functions

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the *intermediate value property.* A function is said to have the **intermediate value property** if it never takes on two values without taking on all the values in between.

Theorem 8 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



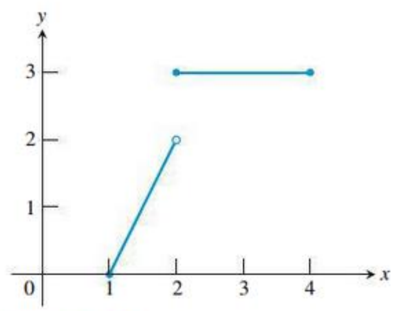


Figure 1.25 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

This theorem may seem so obvious that it does not need to be singled out for special attention. However, it is important because it provides a foundation for the fundamental theorems on which calculus is built. The continuity of f on the interval is essential to Theorem 8. If f is discontinuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 1.25.

A Consequence for Graphing: Connectivity Theorem 8 is the reason why the graph of a function continuous on an interval cannot have any breaks. The graph will be **connected,** a single, unbroken curve, like the graph of sin x. It will not have jumps like those in the graph of the greatest integer function int x, or the separate branches we see in the graph of $1/x$.

Most graphers can plot points *(dot mode)*. Some can turn on pixels between plotted points to suggest an unbroken curve *(connected mode)*. For functions, the connected format basically assumes that outputs *vary continuously* with inputs and do not jump from one value to another without taking on all values in between.

Grapher Failure

In connected mode, a grapher may conceal a function's discontinuities by portraying the graph as a connected curve when it is not. To see what we mean, graph $y = \text{int}(x)$ in a $[-10, 10]$ by $[-10, 10]$ window in both connected and dot modes. A knowledge of where to expect discontinuities will help you recognize this form of grapher failure.

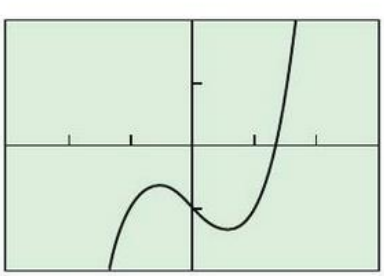
Example 5 Using Theorem 8

Is any real number exactly 1 less than its cube? Compute any such value accurate to three decimal places.

Solution

We answer this question by applying the Intermediate Value Theorem in the following way. Any such number must satisfy the equation $x = x^3 - 1$ or, equivalently, $x^3 - x - 1 = 0$. Hence, we are looking for a zero value of the continuous function $f(x) = x^3 - x - 1$ (Figure 1.26). The function changes sign between 1 and 2, so there must be a point c between 1 and 2 where $f(c) = 0$.

continued



[–3, 3] by [–2, 2]

Figure 1.26 The graph of $f(x) = x^3 - x - 1$. (Example 5)

There are a variety of methods for numerically computing the value of c to be accurate to as many decimal places as your technology allows. For example, a simple application of ZOOM (box) and TRACE using a graphing calculator will quickly give the result of $c \approx 1.324$ accurate to three decimal places. Most calculators have a numerical zero finder that will give an immediate solution as well. *Now Try Exercise 46.*

Can you find the *exact* value of c such that $f(c) = c^3 - c - 1 = 0$? You know this value exists by the application of the Intermediate Value Theorem. Discuss your answer with your classmates and your teacher.

1.3 QUICK REVIEW # 1 - 10

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

1. Find $\lim_{x \to -1} \dfrac{3x^2 - 2x + 1}{x^3 + 4}$.

2. Let $f(x) = \text{int}\, x$. Find each limit or value.

(a) $\lim_{x \to -1^-} f(x)$ **(b)** $\lim_{x \to -1^+} f(x)$ **(c)** $\lim_{x \to -1} f(x)$ **(d)** $f(-1)$

3. Let $f(x) = \begin{cases} x^2 - 4x + 5, & x < 2 \\ 4 - x, & x \geq 2. \end{cases}$

Find each limit or value.

(a) $\lim_{x \to 2^-} f(x)$ **(b)** $\lim_{x \to 2^+} f(x)$ **(c)** $\lim_{x \to 2} f(x)$ **(d)** $f(2)$

In Exercises 4–6, find the remaining functions in the list of functions: $f, g, f \circ g, g \circ f$.

4. $f(x) = \dfrac{2x - 1}{x + 5}$, $g(x) = \dfrac{1}{x} + 1$

5. $f(x) = x^2$, $(g \circ f)(x) = \sin x^2$, domain of $g = [0, \infty)$

6. $g(x) = \sqrt{x - 1}$, $(g \circ f)(x) = 1/x$, $x > 0$

7. Use factoring to solve $2x^2 + 9x - 5 = 0$.

8. Use graphing to solve $x^3 + 2x - 1 = 0$.

In Exercises 9 and 10, let

$$f(x) = \begin{cases} 5 - x, & x \leq 3 \\ -x^2 + 6x - 8, & x > 3. \end{cases}$$

9. Solve the equation $f(x) = 4$.

10. Find a value of c for which the equation $f(x) = c$ has no solution.

1.3 EXERCISES # 1, 11, 19, 23, 25, 33

In Exercises 1–10, find the points of continuity and the points of discontinuity of the function. Identify each type of discontinuity.

1. $y = \dfrac{1}{(x+2)^2}$

2. $y = \dfrac{x+1}{x^2 - 4x + 3}$

3. $y = \dfrac{1}{x^2 + 1}$

4. $y = |x - 1|$

5. $y = \sqrt{2x + 3}$

6. $y = \sqrt[3]{2x - 1}$

7. $y = |x|/x$

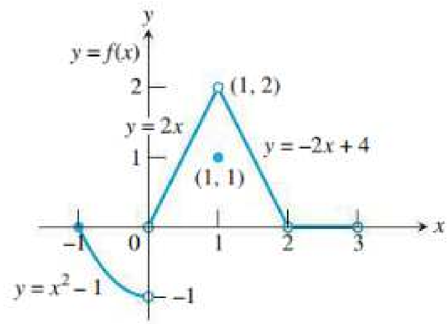
8. $y = \cot x$

9. $y = e^{1/x}$

10. $y = \ln(x + 1)$

In Exercises 11–18, use the function f defined and graphed below to answer the questions.

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$



11. **(a)** Does $f(-1)$ exist?

(b) Does $\lim_{x \to -1^+} f(x)$ exist?

(c) Does $\lim_{x \to -1^+} f(x) = f(-1)$?

(d) Is f continuous at $x = -1$?

12. **(a)** Does $f(1)$ exist?

(b) Does $\lim_{x \to 1} f(x)$ exist?

(c) Does $\lim_{x \to 1} f(x) = f(1)$?

(d) Is f continuous at $x = 1$?

1.3 EXERCISES # 1, 11, 19, 23, 25, 33

13. **(a)** Is f defined at $x = 2$? (Look at the definition of f.)
(b) Is f continuous at $x = 2$?

14. At what values of x is f continuous?

15. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

16. What new value should be assigned to $f(1)$ to make the new function continuous at $x = 1$?

17. **Writing to Learn** Is it possible to extend f to be continuous at $x = 0$? If so, what value should the extended function have there? If not, why not?

18. **Writing to Learn** Is it possible to extend f to be continuous at $x = 3$? If so, what value should the extended function have there? If not, why not?

In Exercises 19–24, **(a)** find each point of discontinuity. **(b)** Which of the discontinuities are removable? not removable? Give reasons for your answers.

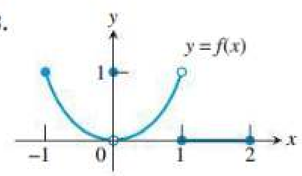
19. $f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2 \end{cases}$

20. $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

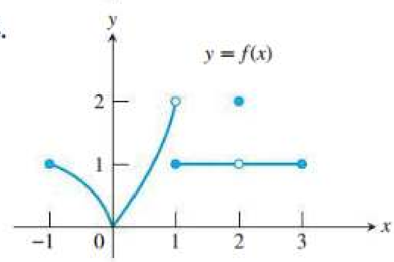
21. $f(x) = \begin{cases} \frac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \geq 1 \end{cases}$

22. $f(x) = \begin{cases} 1 - x^2, & x \neq -1 \\ 2, & x = -1 \end{cases}$

23.



24.



In Exercises 25–30, give a formula for the extended function that is continuous at the indicated point.

25. $f(x) = \frac{x^2 - 9}{x + 3}, \quad x = -3$

26. $f(x) = \frac{x^3 - 1}{x^2 - 1}, \quad x = 1$

27. $f(x) = \frac{\sin x}{x}, \quad x = 0$

28. $f(x) = \frac{\sin 4x}{x}, \quad x = 0$

29. $f(x) = \frac{x - 4}{\sqrt{x} - 2}, \quad x = 4$

30. $f(x) = \frac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}, \quad x = 2$

In Exercises 31 and 32, explain why the given function is continuous.

31. $f(x) = \frac{1}{x - 3}$

32. $g(x) = \frac{1}{\sqrt{x - 1}}$

In Exercises 33–36, use Theorem 7 to show that the given function is continuous.

33. $f(x) = \sqrt{\left(\frac{x}{x + 1}\right)}$

34. $f(x) = \sin(x^2 + 1)$

35. $f(x) = \cos\left(\sqrt[3]{1 - x}\right)$

36. $f(x) = \tan\left(\frac{x^2}{x^2 + 4}\right)$

Group Activity In Exercises 37–40, verify that the function is continuous and state its domain. Indicate which theorems you are using, and which functions you are assuming to be continuous.

37. $y = \frac{1}{\sqrt{x + 2}}$

38. $y = x^2 + \sqrt[3]{4 - x}$

39. $y = |x^2 - 4x|$

40. $y = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

In Exercises 41–44, sketch a possible graph for a function f that has the stated properties.

41. $f(3)$ exists but $\lim_{x \to 3} f(x)$ does not.

42. $f(-2)$ exists, $\lim_{x \to -2^+} f(x) = f(-2)$, but $\lim_{x \to -2} f(x)$ does not exist.

43. $f(4)$ exists, $\lim_{x \to 4} f(x)$ exists, but f is not continuous at $x = 4$.

44. $f(x)$ is continuous for all x except $x = 1$, where f has a nonremovable discontinuity.

45. ***Solving Equations*** Is any real number exactly 1 less than its fourth power? Give any such values accurate to 3 decimal places.

46. ***Solving Equations*** Is any real number exactly 2 more than its cube? Give any such values accurate to 3 decimal places.

47. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

is continuous.

48. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} 2x + 3, & x \leq 2 \\ ax + 1, & x > 2 \end{cases}$$

is continuous.

1.3 EXERCISES # 1, 11, 19, 23, 25, 33

49. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} 4 - x^2, & x < -1 \\ ax^2 - 1, & x \geq -1 \end{cases}$$

is continuous.

50. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} x^2 + x + a, & x < 1 \\ x^3, & x \geq 1 \end{cases}$$

is continuous.

51. **Writing to Learn** Explain why the equation $e^{-x} = x$ has at least one solution.

52. ***Salary Negotiation*** A welder's contract promises a 3.5% salary increase each year for 4 years and Luisa has an initial salary of $36,500.

(a) Show that Luisa's salary is given by

$$y = 36{,}500(1.035)^{\text{int}\, t},$$

where t is the time, measured in years, since Luisa signed the contract.

(b) Graph Luisa's salary function. At what values of t is it continuous?

53. ***Airport Parking*** Valuepark charges $1.10 per hour or fraction of an hour for airport parking. The maximum charge per day is $7.25.

(a) Write a formula that gives the charge for x hours with $0 \leq x \leq 24$. (*Hint:* See Exercise 52.)

(b) Graph the function in part (a). At what values of x is it continuous?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

54. **True or False** A continuous function cannot have a point of discontinuity. Justify your answer.

55. **True or False** It is possible to extend the definition of a function f at a jump discontinuity $x = a$ so that f is continuous at $x = a$. Justify your answer.

56. **Multiple Choice** On which of the following intervals is $f(x) = \dfrac{1}{\sqrt{x}}$ not continuous?

(A) $(0, \infty)$ **(B)** $[0, \infty)$ **(C)** $(0, 2)$
(D) $(1, 2)$ **(E)** $[1, \infty)$

57. **Multiple Choice** Which of the following points is not a point of discontinuity of $f(x) = \sqrt{x - 1}$?

(A) $x = -1$ **(B)** $x = -1/2$ **(C)** $x = 0$
(D) $x = 1/2$ **(E)** $x = 1$

58. **Multiple Choice** Which of the following statements about the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -x + 3, & 1 < x < 2 \end{cases}$$

is not true?

(A) $f(1)$ does not exist.
(B) $\lim_{x \to 0^+} f(x)$ exists.
(C) $\lim_{x \to 2^-} f(x)$ exists.
(D) $\lim_{x \to 1} f(x)$ exists.
(E) $\lim_{x \to 1} f(x) \neq f(1)$

59. **Multiple Choice** Which of the following points of discontinuity of

$$f(x) = \frac{x(x-1)(x-2)^2(x+1)^2(x-3)^2}{x(x-1)(x-2)(x+1)^2(x-3)^3}$$

is not removable?

(A) $x = -1$ **(B)** $x = 0$ **(C)** $x = 1$
(D) $x = 2$ **(E)** $x = 3$

Exploration

60. Let $f(x) = \left(1 + \dfrac{1}{x}\right)^x$.

(a) Find the domain of f. **(b)** Draw the graph of f.

(c) **Writing to Learn** Explain why $x = -1$ and $x = 0$ are points of discontinuity of f.

(d) **Writing to Learn** Is either of the discontinuities in part (c) removable? Explain.

(e) Use graphs and tables to estimate $\lim_{x \to \infty} f(x)$.

Extending the Ideas

61. ***Continuity at a Point*** Show that $f(x)$ is continuous at $x = a$ if and only if

$$\lim_{h \to 0} f(a + h) = f(a).$$

62. ***Continuity on Closed Intervals*** Let f be continuous and never zero on $[a, b]$. Show that either $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

63. ***Properties of Continuity*** Prove that if f is continuous on an interval, then so is $|f|$.

64. ***Everywhere Discontinuous*** Give a convincing argument that the following function is not continuous at any real number.

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

1.4 Rates of Change, Tangent Lines, and Sensitivity

You will be able to use limits to determine instantaneous rates of change, slopes of tangent lines, and sensitivity to change.

- Tangent lines
- Slopes of curves
- Instantaneous rate of change
- Sensitivity

Average Rates of Change

We encounter average rates of change in such forms as average velocity (in miles per hour), growth rates of populations (in percent per year), and average monthly rainfall (in inches per month). The **average rate of change** of a quantity over a period of time is the amount of change divided by the time it takes. In general, the *average rate of change* of a function over an interval is the amount of change divided by the length of the interval.

Example 1 Finding Average Rate of Change

Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

Solution

Since $f(1) = 0$ and $f(3) = 24$, the average rate of change over the interval $[1, 3]$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12.$$

Now Try Exercise 1.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions. Figure 1.27 shows how the number of fruit flies *(Drosophila)* grew in a controlled 50-day experiment. The graph was made by counting flies at regular intervals, plotting a point for each count, and drawing a smooth curve through the plotted points.

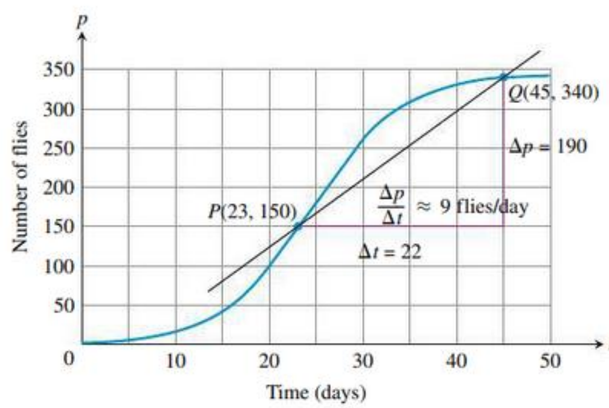


Figure 1.27 Growth of a fruit fly population in a controlled experiment. *Source: Elements of Mathematical Biology.* (Example 2)

Secant to a Curve

A line through two points on a curve is a **secant to the curve.**

Example 2 Growing *Drosophila* in a Laboratory

Use the points $P(23, 150)$ and $Q(45, 340)$ in Figure 1.27 to compute the average rate of change and the slope of the secant line PQ.

Solution

There were 150 flies on day 23 and 340 flies on day 45. This gives an increase of $340 - 150 = 190$ flies in $45 - 23 = 22$ days.

The average rate of change in the population p from day 23 to day 45 was

$$\textit{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day},$$

or about 9 flies per day.

continued

Marjorie Lee Browne (1914–1979)

When Marjorie Browne graduated from the University of Michigan in 1949, she was one of the first two African American women to be awarded a Ph.D. in Mathematics. Browne went on to become chairperson of the mathematics department at North Carolina Central University, and succeeded in obtaining grants for retraining high school mathematics teachers.

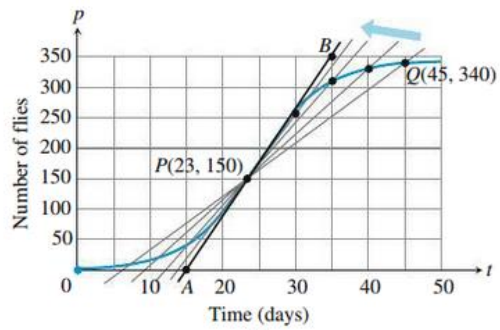
This average rate of change is also the slope of the secant line through the two points P and Q on the population curve. We can calculate the slope of the secant PQ from the coordinates of P and Q.

$$\textit{Secant slope:}\quad \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}$$

Now Try Exercise 7.

As suggested by Example 2, *we can always think of an average rate of change as the slope of a secant line.*

In addition to knowing the average rate at which the population grew from day 23 to day 45, we may also want to know how fast the population was growing on day 23 itself. To find out, we can watch the slope of the secant PQ change as we back Q along the curve toward P. The results for four positions of Q are shown in Figure 1.28.



(a)

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$(340 - 150)/(45 - 23) \approx 8.6$
(40, 330)	$(330 - 150)/(40 - 23) \approx 10.6$
(35, 310)	$(310 - 150)/(35 - 23) \approx 13.3$
(30, 265)	$(265 - 150)/(30 - 23) \approx 16.4$

(b)

Figure 1.28 (a) Four secants to the fruit fly graph of Figure 1.27, through the point $P(23, 150)$. (b) The slopes of the four secants.

In terms of geometry, what we see as Q approaches P along the curve is this: The secant PQ approaches the tangent line AB that we drew by eye at P. This means that within the limitations of our drawing, the slopes of the secants approach the slope of the tangent, which we calculate from the coordinates of A and B to be

$$\frac{350 - 0}{35 - 15} = 17.5 \text{ flies/day}.$$

In terms of population, what we see as Q approaches P is this: The average growth rates for increasingly smaller time intervals approach the slope of the tangent to the curve at P (17.5 flies per day). The slope of the tangent line is therefore the number we take as the rate at which the fly population was growing on day $t = 23$.

Tangent to a Curve

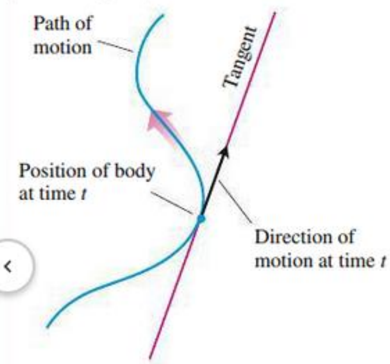
The moral of the fruit fly story would seem to be that we should define the rate at which the value of the function $y = f(x)$ is changing with respect to x at any particular value $x = a$ to be the slope of the tangent to the curve $y = f(x)$ at $x = a$. But how are we to define the tangent line at an arbitrary point P on the curve and find its slope from the formula $y = f(x)$? The problem here is that we know only one point. Our usual definition of slope requires two points.

The solution that mathematician Pierre Fermat found in 1629 proved to be one of that century's major contributions to calculus. We still use his method of defining tangents to produce formulas for slopes of curves and rates of change:

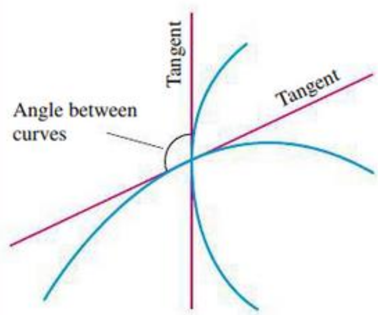
1. We start with what we can calculate, namely, the slope of a secant through P and a point Q nearby on the curve.

Why Find Tangents to Curves?

In mechanics, the tangent determines the direction of a body's motion at every point along its path.



In geometry, the tangents to two curves at a point of intersection determine the angle at which the curves intersect.



In optics, the tangent determines the angle at which a ray of light enters a curved lens (more about this in Section 3.2). The problem of how to find a tangent to a curve became the dominant mathematical problem of the early 17th century, and it is hard to overstate how badly the scientists of the day wanted to know the answer. Descartes went so far as to say that the problem was the most useful and most general problem not only that he knew but that he had any desire to know.

2. We find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.

3. We define the *slope of the curve* at P to be this number and define the *tangent to the curve* at P to be the line through P with this slope.

Example 3 Finding Slope and Tangent Line

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution

We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$ on the curve (Figure 1.29). Note that h can be positive or negative, but it cannot be 0 because we need two distinct points.

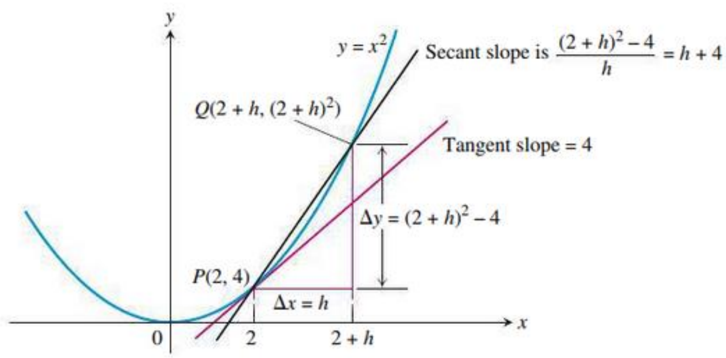


Figure 1.29 The slope of the tangent to the parabola $y = x^2$ at $P(2, 4)$ is 4.

We then write an expression for the slope of the secant line and find the limiting value of this slope as Q approaches P along the curve.

$$\begin{aligned}\text{Secant slope} &= \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{h} \\ &= \frac{4 + 4h + h^2 - 4}{h} \\ &= \frac{4h + h^2}{h} = 4 + h\end{aligned}$$

The slope of the tangent line must be less than $4 + h$ for every positive value of h and must be greater than $4 + h$ for every negative value of h. The only number that satisfies these bounds is 4. Therefore, the limit of the secant slope as Q approaches P along the curve is

$$\lim_{Q \to P} (\text{secant slope}) = \lim_{h \to 0} (4 + h) = 4.$$

Thus, the slope of the parabola at P is 4.

The tangent to the parabola at P is the line through $P(2, 4)$ with slope $m = 4$.

$$y = 4 + 4(x - 2)$$

Now Try Exercise 11 (a, b).

Pierre de Fermat (1601–1665)

Fermat was a lawyer who worked for the regional parliament in Toulouse, France. He invented his method of tangents in 1629 to solve the problem of finding the cone of greatest volume that can be formed by cutting a sector out of a circle and then folding the circle to join the cut edges (see Exploration 1, Section 4.4, page 233). Fermat shares with René Descartes the honor of having invented what today we know as the Cartesian coordinate system, enabling us to express algebraic equations graphically and so connect algebra and geometry. Fermat made many contributions to the development of calculus, but he did not publish his discoveries. His mathematical writing was largely confined to professional correspondence and papers written for personal friends.

Slope of a Curve

To find the tangent to a curve $y = f(x)$ at a point $P(a, f(a))$, we use the same dynamic procedure. We calculate the slope of the secant line through P and a point $Q(a + h, f(a + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 1.30).

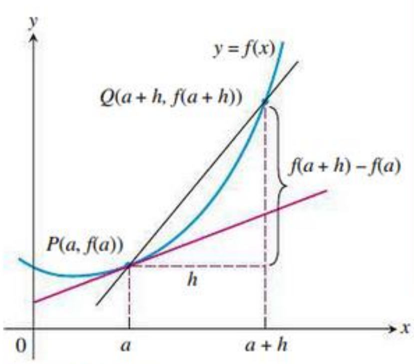


Figure 1.30 The tangent slope is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

If the limit exists, it is the slope of the curve at P, and we define the tangent at P to be the line through P having this slope.

Definition Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

The **tangent line to the curve** at P is the line through P with this slope.

Example 4 Exploring Slope and Tangent

Let $f(x) = 1/x$.

(a) Find the slope of the curve at $x = a$.

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ for different values of a?

Solution

(a) The slope at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h \to 0} \cdot \frac{-h}{ha(a+h)}$$

$$= \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

(b) The slope will be $-1/4$ if

$$-\frac{1}{a^2} = -\frac{1}{4}$$

$$a^2 = 4 \quad \text{Multiply by } -4a^2.$$

$$a = \pm 2.$$

The curve has the slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 1.31).

(c) The slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep. We see this again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent becomes increasingly horizontal. ***Now Try Exercise 19.***

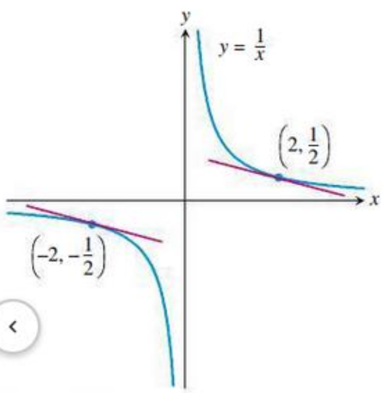


Figure 1.31 The two tangent lines to $y = 1/x$ having slope $-1/4$. (Example 4)

All of These Are the Same

1. the slope of $y = f(x)$ at $x = a$
2. the slope of the tangent to $y = f(x)$ at $x = a$
3. the (instantaneous) rate of change of $f(x)$ with respect to x at $x = a$
4. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

An Alternate Form

In Chapter 2, we will introduce the expression

$$\frac{f(x) - f(a)}{x - a}$$

as an important and useful alternate form of the **difference quotient of *f* at *a*.** (See Exercise 57.)

The expression

$$\frac{f(a+h) - f(a)}{h}$$

is the **difference quotient of *f* at *a*.** Suppose the difference quotient has a limit as h approaches zero. If we interpret the difference quotient as a secant slope, the limit is the slope of both the curve and the tangent to the curve at the point $x = a$. If we interpret the difference quotient as an average rate of change, the limit is the function's rate of change with respect to x at the point $x = a$. This limit is one of the two most important mathematical objects considered in calculus. We will begin a thorough study of it in Chapter 2.

About the Word *Normal*

When analytic geometry was developed in the 17th century, European scientists still wrote about their work and ideas in Latin, the one language that all educated Europeans could read and understand. The Latin word *normalis,* which scholars used for *perpendicular,* became *normal* when they discussed geometry in English.

Normal to a Curve

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

Example 5 Finding a Normal Line

Write an equation for the normal to the curve $f(x) = 4 - x^2$ at $x = 1$.

Solution

The slope of the tangent to the curve at $x = 1$ is

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{4 - (1+h)^2 - 3}{h}$$
$$= \lim_{h\to 0} \frac{4 - 1 - 2h - h^2 - 3}{h}$$
$$= \lim_{h\to 0} \frac{-h(2+h)}{h} = -2.$$

Thus, the slope of the normal is $1/2$, the negative reciprocal of -2. The normal to the curve at $(1, f(1)) = (1, 3)$ is the line through $(1, 3)$ with slope $m = 1/2$.

$$y = 3 + \frac{1}{2}(x - 1)$$

You can support this result by drawing the graphs in a square viewing window.

Now Try Exercise 11 (c, d).

Particle Motion

In this chapter, we have considered only objects moving in one direction. In Chapter 2, we will deal with more complicated motion.

Velocity Revisited

The function $y = 16t^2$ that gave the distance fallen by the rock in Example 1, Section 1.1, was the rock's *position function.* A body's average velocity along a coordinate axis (here, the y-axis) for a given period of time is the average rate of change of its *position* $y = f(t)$. Its *instantaneous velocity* at any time t is the *instantaneous rate of change* of position with respect to time at time t, or

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h}.$$

We saw in Example 1, Section 1.1, that the rock's instantaneous velocity at $t = 2$ sec was 64 ft/sec.

Example 6 Finding Instantaneous Rate of Change

Find

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

for the function $f(t) = 2t^2 - 1$ at $t = 2$. Interpret the answer if $f(t)$ represents a position function in feet of an object at time t seconds.

continued

Solution

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \to 0} \frac{2(2+h)^2 - 1 - (2 \cdot 2^2 - 1)}{h}$$
$$= \lim_{h \to 0} \frac{8 + 8h + 2h^2 - 1 - 7}{h}$$
$$= \lim_{h \to 0} \frac{8h + 2h^2}{h}$$
$$= \lim_{h \to 0} (8 + 2h) = 8$$

The instantaneous rate of change of the object is 8 ft/sec. ***Now Try Exercise 23.***

Example 7 Investigating Free Fall

Find the velocity of the falling rock in Example 1, Section 1.1, at $t = 1$ sec.

Solution

The position function of the rock is $f(t) = 16t^2$. The average velocity of the rock over the interval between $t = 1$ and $t = 1 + h$ sec was

$$\frac{f(1+h) - f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's velocity at the instant $t = 1$ was

$$\lim_{h \to 0} 16(h + 2) = 32 \text{ ft/sec}.$$

Now Try Exercise 31.

Sensitivity

We live in an interconnected world where changes in one quantity cause changes in another. For example, crop yields per acre depend on rainfall. If rainfall has been low, each small increase in the amount of rain creates a small increase in crop yield. For a drug that works to lower a patient's temperature, each small increase in the amount of the drug will lower the temperature a small amount. The mathematical connection between such changes is known as **sensitivity.** Sensitivity describes how one variable responds to small changes in another variable.

How much the patient's temperature drops depends not just on how much more of the drug is given, but also on how large the dose already is. Sensitivity changes as the dosage changes. If we let T denote the patient's temperature and D the dosage, then the sensitivity is given by

$$\text{sensitivity} = \lim_{\Delta D \to 0} \frac{\Delta T}{\Delta D}.$$

If we know the sensitivity and we have a very small change in the dosage, then we can approximate the change in the patient's temperature by

$$\Delta T \approx \text{sensitivity} \times \Delta D.$$

Example 8 Measuring Sensitivity to Medicine

A patient enters the hospital with a temperature of 102°F and is given medicine to lower the temperature. As a function of the dosage, D, measured in milligrams, the patient's temperature will be

$$T(D) = 99 + \frac{3}{1 + D}.$$

Find and interpret the sensitivity of the patient's temperature to the medicine dosage when $D = 1$ mg.

continued

Solution

Let h be the increase in the dosage, $\Delta D = h$, so that the sensitivity is

$$\lim_{h \to 0} \frac{\Delta T}{h} = \lim_{h \to 0} \frac{99 + \dfrac{3}{1 + (1 + h)} - \left(99 + \dfrac{3}{1 + 1}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{3}{2 + h} - \dfrac{3}{2}}{h} = \lim_{h \to 0} \frac{6 - 6 - 3h}{2(2 + h)h} = \frac{-3}{4} \text{ degrees per mg.}$$

This means that when the dosage is 1 mg, a small additional dosage, ΔD mg, will result in a drop in the patient's temperature of approximately $3/4\ \Delta D$ degrees.

Now Try Exercise 37.

1.4 QUICK REVIEW # 1 – 10

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1 and 2, find the increments Δx and Δy from point A to point B.

1. $A(-5, 2)$, $B(3, 5)$ **2.** $A(1, 3)$, $B(a, b)$

In Exercises 3 and 4, find the slope of the line determined by the points.

3. $(-2, 3)$, $(5, -1)$ **4.** $(-3, -1)$, $(3, 3)$

In Exercises 5–9, write an equation for the specified line.

5. through $(-2, 3)$ with slope $= 3/2$

6. through $(1, 6)$ and $(4, -1)$

7. through $(1, 4)$ and parallel to $y = -\frac{3}{4}x + 2$

8. through $(1, 4)$ and perpendicular to $y = -\frac{3}{4}x + 2$

9. through $(-1, 3)$ and parallel to $2x + 3y = 5$

10. For what value of b will the slope of the line through $(2, 3)$ and $(4, b)$ be $5/3$?

1.4 EXERCISES # 1, 7, 9, 13, 15, 19, 23, 27,33, 35, 37

In Exercises 1–6, find the average rate of change of the function over each interval.

1. $f(x) = x^3 + 1$
(a) $[2, 3]$ (b) $[-1, 1]$

2. $f(x) = \sqrt{4x + 1}$
(a) $[0, 2]$ (b) $[10, 12]$

3. $f(x) = e^x$
(a) $[-2, 0]$ (b) $[1, 3]$

4. $f(x) = \ln x$
(a) $[1, 4]$ (b) $[100, 103]$

5. $f(x) = \cot x$
(a) $[\pi/4, 3\pi/4]$ (b) $[\pi/6, \pi/2]$

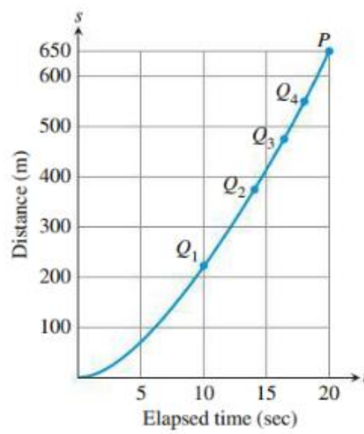
6. $f(x) = 2 + \cos x$
(a) $[0, \pi]$ (b) $[-\pi, \pi]$

In Exercises 7 and 8, a distance-time graph is shown.

(a) Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table. What is the appropriate unit for these slopes?

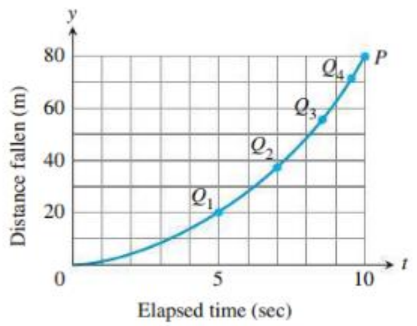
(b) Estimate the speed at point P.

7. ***Accelerating from a Standstill*** The figure shows the distance-time graph for a 1994 Ford® Mustang Cobra™ accelerating from a standstill.



1.4 EXERCISES # 1, 7, 9, 13, 15, 19, 23, 27,33, 35, 37

8. ***Lunar Data*** The accompanying figure shows a distance-time graph for a wrench that fell from the top platform of a communication mast on the moon to the station roof 80 m below.



In Exercises 9–12, at the indicated point find

(a) the slope of the curve,

(b) an equation of the tangent, and

(c) an equation of the normal.

(d) Then draw a graph of the curve, tangent line, and normal line in the same square viewing window.

9. $y = x^2$ at $x = -2$ **10.** $y = x^2 - 4x$ at $x = 1$

11. $y = \dfrac{1}{x-1}$ at $x = 2$ **12.** $y = x^2 - 3x - 1$ at $x = 0$

In Exercises 13 and 14, find the slope of the curve at the indicated point.

13. $f(x) = |x|$ at **(a)** $x = 2$ **(b)** $x = -3$

14. $f(x) = |x - 2|$ at $x = 1$

In Exercises 15–18, determine whether the curve has a tangent at the indicated point. If it does, give its slope. If not, explain why not.

15. $f(x) = \begin{cases} 2 - 2x - x^2, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$ at $x = 0$

16. $f(x) = \begin{cases} -x, & x < 0 \\ x^2 - x, & x \geq 0 \end{cases}$ at $x = 0$

17. $f(x) = \begin{cases} 1/x, & x \leq 2 \\ \dfrac{4-x}{4}, & x > 2 \end{cases}$ at $x = 2$

18. $f(x) = \begin{cases} \sin x, & 0 \leq x < 3\pi/4 \\ \cos x, & 3\pi/4 \leq x \leq 2\pi \end{cases}$ at $x = 3\pi/4$

In Exercises 19–22, **(a)** find the slope of the curve at $x = a$.
(b) **Writing to Learn** Describe what happens to the tangent at $x = a$ as a changes.

19. $y = x^2 + 2$

20. $y = 2/x$

21. $y = \dfrac{1}{x-1}$

22. $y = 9 - x^2$

Find the instantaneous rate of change of the position function $y = f(t)$ in feet at the given time t in seconds.

23. $f(t) = 3t - 7, t = 1$

24. $f(t) = 3t^2 + 2t, \quad t = 3$

25. $f(t) = \dfrac{t+1}{t}, \quad t = 2$

26. $f(t) = t^3 - 1, \quad t = 2$

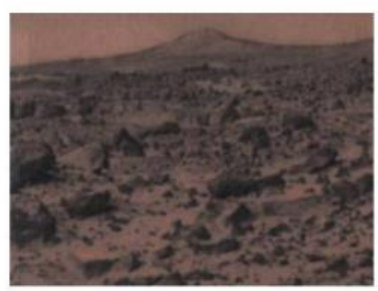
27. ***Free Fall*** An object is dropped from the top of a 100-m tower. Its height above ground after t sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

28. ***Rocket Launch*** At t sec after lift-off, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing after 10 sec?

29. ***Area of Circle*** What is the rate of change of the area of a circle with respect to the radius when the radius is $r = 3$ in.?

30. ***Volume of Sphere*** What is the rate of change of the volume of a sphere with respect to the radius when the radius is $r = 2$ in.?

31. ***Free Fall on Mars*** The equation for free fall at the surface of Mars is $s = 1.86t^2$ m with t in seconds. Assume a rock is dropped from the top of a 200-m cliff. Find the speed of the rock at $t = 1$ sec.



32. ***Free Fall on Jupiter*** The equation for free fall at the surface of Jupiter is $s = 11.44t^2$ m with t in seconds. Assume a rock is dropped from the top of a 500-m cliff. Find the speed of the rock at $t = 2$ sec.

33. ***Horizontal Tangent*** At what point is the tangent to $f(x) = x^2 + 4x - 1$ horizontal?

34. ***Horizontal Tangent*** At what point is the tangent to $f(x) = 3 - 4x - x^2$ horizontal?

1.4 EXERCISES # 1, 7, 9, 13, 15, 19, 23, 27,33, 35, 37

35. *Finding Tangents and Normals*

(a) Find an equation for each tangent to the curve $y = 1/(x - 1)$ that has slope -1. (See Exercise 21.)

(b) Find an equation for each normal to the curve $y = 1/(x - 1)$ that has slope 1.

36. *Finding Tangents* Find the equations of all lines tangent to $y = 9 - x^2$ that pass through the point $(1, 12)$.

37. *Sensitivity* A patient's temperature T as a function of the dosage D of a medicine is given by $T(D) = 99 + 4/(1 + D)$. Find and interpret the sensitivity of the patient's temperature to the dosage when $D = 2$ mg.

38. *Sensitivity* If a ball is thrown straight up with an initial velocity of v feet per second, it will reach a maximum height of $H = v^2/64$ feet. Find and interpret the sensitivity of the height to the initial velocity when the initial velocity is 40 ft/sec.

39. Table 2.2 gives the total amount of all U.S. exported wheat in millions of bushels for several years.

TABLE 2.2 U.S. Exported Wheat

Year	Exported Wheat (millions of bushels)
2008	1015
2009	879
2010	1291
2011	1051
2012	1007
2013	900

Source: U.S. Department of Agriculture, Economic Research Service, Wheat Data, Table 21.

(a) Make a scatter plot of the data in the table.

(b) Let P represent the point corresponding to 2008, Q_1 the point corresponding to 2011, Q_2 the point corresponding to 2012, and Q_3 the point corresponding to 2013. Find the slope of the secant line PQ_i for $i = 1, 2, 3$.

40. Table 2.3 gives the amount of federal spending in billions of dollars for national defense for several years.

TABLE 2.3 National Defense Spending

Year	National Defense Spending ($ billion)
2008	616
2009	661
2010	693
2011	706
2012	678
2013	633

Source: U.S. Office of Management and Budget, *Budget Authority by Function and Subfunction, Outlay by Function and Subfunction,* Table 492.

(a) Find the average rate of change in spending from 2008 to 2013.

(b) Find the average rate of change in spending from 2008 to 2011.

(c) Find the average rate of change in spending from 2011 to 2013.

(d) Writing to Learn Explain why someone might be hesitant to make predictions about the rate of change of national defense spending based on the data given in Table 2.3.

Standardized Test Questions

41. True or False If the graph of a function has a tangent line at $x = a$, then the graph also has a normal line at $x = a$. Justify your answer.

42. True or False The graph of $f(x) = |x|$ has a tangent line at $x = 0$. Justify your answer.

43. Multiple Choice If the line L tangent to the graph of a function f at the point $(2, 5)$ passes through the point $(-1, -3)$, what is the slope of L?

(A) $-3/8$ **(B)** $3/8$ **(C)** $-8/3$ **(D)** $8/3$ **(E)** undefined

44. Multiple Choice Find the average rate of change of $f(x) = x^2 + x$ over the interval $[1, 3]$.

(A) -5 **(B)** $1/5$ **(C)** $1/4$ **(D)** 4 **(E)** 5

45. Multiple Choice Which of the following is an equation of the tangent to the graph of $f(x) = 2/x$ at $x = 1$?

(A) $y = -2x$ **(B)** $y = 2x$ **(C)** $y = -2x + 4$

(D) $y = -x + 3$ **(E)** $y = x + 3$

46. Multiple Choice Which of the following is an equation of the normal to the graph of $f(x) = 2/x$ at $x = 1$?

(A) $y = \frac{1}{2}x + \frac{3}{2}$ **(B)** $y = -\frac{1}{2}x$ **(C)** $y = \frac{1}{2}x + 2$

(D) $y = -\frac{1}{2}x + 2$ **(E)** $y = 2x + 5$

Explorations

In Exercises 47 and 48, complete the following for the function.

(a) Compute the difference quotient

$$\frac{f(1 + h) - f(1)}{h}.$$

(b) Use graphs and tables to estimate the limit of the difference quotient in part (a) as $h \to 0$.

(c) Compare your estimate in part (b) with the given number.

(d) Writing to Learn Based on your computations, do you think the graph of f has a tangent at $x = 1$? If so, estimate its slope. If not, explain why not.

47. $f(x) = e^x$, e **48.** $f(x) = 2^x$, $\ln 4$

1.4 EXERCISES # 1, 7, 9, 13, 15, 19, 23, 27,33, 35, 37

Group Activity In Exercises 49–52, the curve $y = f(x)$ has a **vertical tangent** at $x = a$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \infty$$

or if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = -\infty.$$

In each case, the right- and left-hand limits are required to be the same: both $+\infty$ or both $-\infty$.

Use graphs to investigate whether the curve has a vertical tangent at $x = 0$.

49. $y = x^{2/5}$

50. $y = x^{3/5}$

51. $y = x^{1/3}$

52. $y = x^{2/3}$

Extending the Ideas

In Exercises 53 and 54, determine whether the graph of the function has a tangent at the origin. Explain your answer.

53. $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

54. $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

55. ***Sine Function*** Estimate the slope of the curve $y = \sin x$ at $x = 1$. (*Hint:* See Exercises 47 and 48.)

56. Consider the function f given in Example 1. Explain how the average rate of change of f over the interval $[3, 3 + h]$ is the same as the difference quotient of f at $a = 3$.

57. (a) Let $x = a + h$. Show algebraically how the difference quotient of f at a,

$$\frac{f(a+h) - f(a)}{h},$$

is equivalent to an alternate form given by

$$\frac{f(x) - f(a)}{x - a}.$$

(b) Writing to Learn Why do you think we discuss two forms of the difference quotient of f at a?